ΘΕΩΝΟΣ ΣΜΥΡΝΑΙΟΥ

MATHEMATICS USEFUL FOR UNDERSTANDING PLATO

by
THEON OF SMYRNA
Platonic Philosopher

Translated from
the 1892 Greek/French edition of J. Dupuis
by
ROBERT and DEBORAH LAWLOR
and
edited and annotated by Christos Toulis
and others.

with
an appendix of notes by Dupuis,
a copious glossary, index of works etc.

SECRET DOCTRINE REFERENCE SERIES

WIZARDS BOOKSHELF
SAN DIEGO
1979

LIBRARY OF CONGRESS CATALOG CARD NUMBER 77-73716
INTERNATIONAL STANDARD BOOK NUMBER 0-913510-24-6

WIZARDS BOOKSHELF
BOX 6600, SAN DIEGO,
CA 92106
USA

6-16-79

TABLE OF CONTENTS

NOTES by J. Dupuis

Glossary

Antiphony, Ἀντίφωνια (Voice against voice), accord of two voices with the octave or the double octave.

Bomisque, Βωμισκος (small altar), rectangular parallelopiped having three unequal sides.

Dadouchia, - Dadououchos (from Δᾷς torch, Εχειν to have), procession with flaming torches: During the ceremonies of initiation to the mysteries.

Deficient number, a number, the sum of whose fractional parts is less than the number itself.

Diagram from Διάγραμμα, table or model giving a general survey of all the sounds of a system.

Diesis, from Δίεσις, the smallest interval in each type: consequently a quarter-tone in the enharmonic type and a half-tone in the other two (the diatonic and the chromatic).

Ditone, from Διτόνον, the interval of two undivided tones.

Docide from Δοκίς Rectangular Parallelopiped—(from Gr parallelepipedos = Having parallel sides or two parallel planes) having two equal sides and a larger third side.

Epimer, Ἐπιμερής relationship: relationship in the form of

$$1 + \frac{m}{m + n}$$

Epitrite from Ἐπίτριτος or sesquitertian relationship: relationship of 4 to 3, measuring the consonance of the fourth.

Euthemetric Εὐθυμετρικος number: a number representing a linear measure only, that is a prime number.

Hemiole from Ἡμιόλιον or sesquialter relationship: the relationship of 3 to 2. It measures the consonance of the fifth.

Heteromecic (Fr) Ἑτερόμήκης unequilateral number: the product of two factors which differ by one unit.

Hierophanty — Ἱεροφαντεια (from Ἱερός sacred, and Φαινο to reveal): explanation of the mysteries, one of the ceremonies of initiation.

Hypate, - Ὕπατος implies the last — but also the first, or highest. Tetracord of the hypates: the lowest of the tetracords of the perfect system. The hypate of the bypates was the lowest string of the tetracord of the hypates; it was higher by one tone than the proslambanomenos. The hypate of the meses was the lowest of the tetracord of the meses and also served as the highest of the

tetracord of the hypates.

Hypepimer - Ὑποεπιμερής relationship: the inverse relationship of the epimer relationship.

Hyperbole - Ὑπερβολιος. Tetracord of the hyperboles: the highest of the tetracords in the perfect system.

Hyperhypate - Ὑπερυπάται the string above the parhypate of the hypates and below the hypate of the meses.

Hypo-polyepimer - Ὑποπολυεπιμερής relationship: the inverse of the polyepimer relationship.

Leimma - Λεῖμμα (also surplus) excess of the fourth over the double tone.

Lichanos, (index finger) the string plucked by the index finger or the sound emanating from this string: the string indicating the type (diatonic, chromatic or enharmonic). Lichanos of the hypates: the same as the one above the hyperhypate. Lichanos of the meses: string below the meses.

Mean, Μεσότης: proportion formed of three numbers so that the excess of the first over the second is to the excess of the second over the third, as the first is to itself, to the second or to the third, or as the second is to the third, or inversely.

Mese, Μέσος – Middle, string so named because, in the perfect system, it is at the distance of an octave from the extremes (the proslambanomenos and the nete of the hyperbolaion).

Multisuperpartial, fractionary number in the form of $a + 1/m$

Nete, Νήτη: last string, in going up, of each of the two last tetracords of the perfect system.

Octacord, Ὀκταχορδος Λύρα: eight-stringed lyre commonly attributed to Pythagoras. It includes two disjunct tetracords (that is, tetracords separated by one tone).

Paramese, Παράμεσος (next to the middle), string adjacent to the mese.

Paraphony, Παραφωνια:the resultant consonance, like the fourth and the fifth, of two sounds which are neither in unison nor an octave apart.

Parhypate, Παρυπάτη: string adjacent to the hypate.

Plinthe, Πλινθίς: a rectangular parallelopiped having two equal sides and the third smaller.

Polyepimer Πολυεπιμερής, fractionary number in the form

$$a + \frac{m}{m+n}$$

Promecic' Προμήκης number: the product of two different numbers.

Proslambanomenos: an added string, giving the lowest sound of the perfect system.

Sesquioctave, sesquiotavus, an eighth in addition to the unity, that is $1 + 1/8$.

Sesquipartial or superpartial, superpartians, relationship containing one part in addition to the unity, that is to say in the form $1 + 1/m$.

Sesquiquartan, sesquiquartus; a fourth in addition to the unity, that is to say $1 + 1/4$.

Sesquiquintan, a fifth in addition to the unity, that is to say $1 + 1/5$.

Sesquitertian, sesquitertius, a third in addition to the unity, that is $1 + 1/3$.

Sub-sesquioctave: the relationship of 8 to 9, the inverse of the sesquioctave relationship 9 to 8.

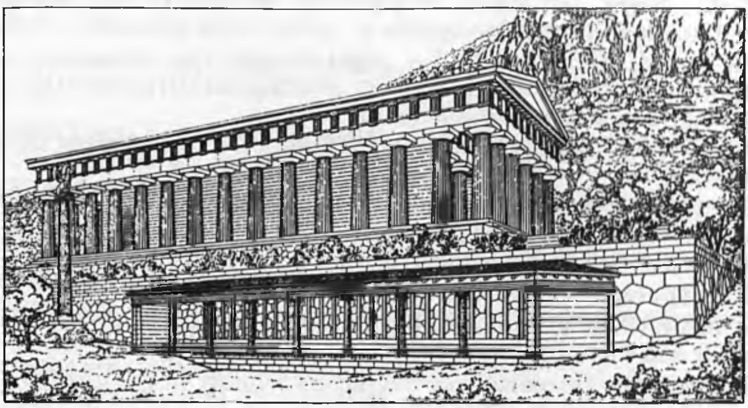
Sub-sesquipartial : the inverse of the sesquipartial relationship

Sub-sesquiquartan, the relationship of 4 to 5, the inverse of the sesquiquartan relationship $5/4$

Superpartial, see sesquipartial.

Trihemitone, Τριημιτονον: interval of one and a half tones, undivided.

Trite, Τριτη implies third: third string of the tetracord of the hyperbolaion and of the disjunct tetracord, going from high to low.



PUBLISHERS NOTE

This is the twenty-first in the Secret Doctrine Reference Series, and the first translation into English of the known works of Theon of Smyrna. The Secret Doctrine is our primary motivating force stimulating inquiry into the ancient philosophies; they allowed man a greater dignity through his understanding and identity with nature. In the course of this project a number of factors have presented themselves, some of which should be mentioned.

J. Dupuis' Greek/French edition, published in Paris, 1892, is the basis for our translation. It consisted of the Greek text on the left hand pages, and Dupuis French translation with footnotes on the right hand pages. Over 400 pages in all. We regret that current modern economics prevents the inclusion of the original Greek text, a factor which would double the size and cost of the book, and thus reduce exposure. In addition to the text, in this edition are translated Dupuis review of earlier editions, his footnotes where useful, his appendix of notes, and his explanatory index. It was found that his French translation though good, did have some errors.

Scholars may discover that there are discrepancies, or variation in terms used in this translation when compared with earlier studies. This is due in part to the series of people contributing comments and suggestions to help clarify the text, or render it more accurately. The work of translation was not done with an eye toward conformity with others, but hopefully to present a lucid and accurate edition. As an instance, in *Greek Mathematics,* Sir Thomas Heath credits Theon of Smyrna with 'two important things not found in Nichomachus' (p.70-71). Heath's 'side and diameter numbers,' are to be seen as *'lateral and diagonal numbers'* on pages 29 and 30 of the present edition. Also Heath's summary . . . 'if m^2 is a square number, either m^2 or m^2-1 is divisible by 3, and again, either m^2 or m^2-1 is divisible by 4.' *'Greek Math* p. 71) . . . is shown in the present edition as *'in the triple progression,'* on p. 24, second paragraph. It is not unlikely that there are variations in the many manuscripts attributed to Theon, and this may be an additional cause of differences be-

tween renditions.

The problems of translation. Occasionally, Greek terms will not transliterate correctly into English. English is preiminently a language of *things,* of possessions. Greek, however, seems better able to embrace philosophy, and *ideas,* and is perhaps a more spiritual language. There is for instance, a great difficulty in the Greek word Μεσότης, variously rendered as *mean, median,* and by Dupuis in his French as *médiaté*. They are all inadequate. It is one of the most fundamental concepts necessary for grasping Theon's work, and is directly related to the concept of *the one,* or *unity,* and the *monad.* *Μέσος* stands for equipoise more than average, the point of balance or node, more than merely 'halfway.' In its way, it is the unseen fulcrum of nature that exists only because of *observable* opposite polarities on either side of it, forcing the philosophical necessity of the monas, or esoteric cause, between. In this translation *mean* has been used in most cases, *median* in some, with the *monad* its parent concept. Theon, in concluding his proofs to account for physical observations sometimes says: 'thus the phenomena are saved.' Of course this is a literal translation, and it may also be rendered as 'explained,' being that the proof in his theorem justifies, or accounts for the observation adequately.

For further reading, the works of Thomas Taylor (1758-1831) offer a grasp of Platonic philosophy that is yet to be demonstrated by the plethora of dead letter grammarians who have followed him. J. Ralston Skinner's *Source of Measures*, 1876, (rpr, 1972) gives insights to natural numbers 6561, 20612, and 5184, as used in the Gt Pyramid based on π. Also, a seldom seen title is Urwick's *Message of Plato*, and for understanding the concept of the monad, Leibnitz *Monadology*, Blavatsky's *Secret Doctrine*, and Dr. J.G. Macvicar's *Sketch of a Philosophy*, London, 1869, will lead one toward understanding. Lastly, it should be considered that the publisher considers this edition as a first step, and hopes that it will offer sufficient encouragement to serious students, to generate constructive comments. Thus may the future hold greater understanding, and men, having right thoughts, will give rise to right actions, and the whole will be improved by dint of its parts.

San Diego, 1979

INTRODUCTION

The emergence of a creative as well as an academic reinvestigation by Pythagorean philosophy has prompted this English translation and republication of the works of Theon of Smyrna. This work appears to have been a text book intended for students who were beginning a study of the works of Plato. In its original form there were five sections: 1) Arithmetic 2) Plane Geometry 3) Stereometry (solid geometry) 4) Music 5) Astronomy. Sections 2 and 3 on Geometry have been lost while the others remain in their entirety and are presented here.

Theon lived in the early part of the second century A.D. Almost nothing is known of his life but he is sometimes referred to as Theon the Ancient by Greek historians to prevent any confusion with Theon of Alexandria, a 4th century Greek mathematician. Theon of Smyrna was most certainly a contemporary of Nichomachus Gerasa (Palestine) whose famous work, *Introduction to Arithmetic* must have appeared about the same time. Although these works have some remarkable similarity, it can be conjectured that these two authors were not aware of each other's work but were undoubtedly drawing upon the same ancient resource material. The *Introduction to Arithmetic* by Nichomachus relies more on expositions using schematic numerical tables and lengthy explanations. The author himself seems to have been more influenced by Eclectic and Aristotelian procedures of thought than was Theon and perhaps for these reasons "The Introduction" was selected to become the classic text of Greek arithmetic by European culture. Nichomachus' work was copied and translated and republished many times in a great number of different languages and was in common use as a basic mathematical text for a long period in Gothic and Renaissance Europe while Theon's book was only completely translated into one modern European language — French — in the 1892 edition of J. Dupuis, which was utilized for the present English translation, although there were some excerpts translated into English in Thomas Taylor's *The Theoretic*

Arithmetic of the Pythagoreans. Theon's work with shorter explanations covers many more arithmetic topics and is considered to demonstrate that he was the more developed mathematician of the two. Taking into account the separate *Introduction* which Nichomachus wrote on Music and Astronomy, it can be said that Theon was able to do in one book what Nichomachus did in four. Many propositions omitted in Nichomachus are found in Theon's work, especially in topics which draw a relationship between geometry and number, such as the treatment of polygonal numbers, the special properties of square and pyramidal numbers, root ratios, point, line, plane and solid figure development, definitions of gnomons, varieties of solids, and many others. Probably the most important concept omitted by Nichomachus and handled directly by Theon is the concept of the Monad which is the core of the philosophical and esoteric doctrine which underlies these numerical expositions.

Theon also reveals the names of a number of his sources and of associates involved in Pythagorean pursuits: Archytas, Aristotle, Adrastus, Philolaus, Plato, Eudoxus, Hipparchus, Thrasyllus, Hippasus and others from whom he draws his material. These and other considerations allow the speculation that Theon was an advanced Pythagorean, and that this work carries an effective impulse of that tradition.

Briefly we may say that this tradition which Theon is transmitting to us distinguishes itself through its unfaltering integration of numerical or scientific modes of knowledge with metaphysical and mystic cosmologies. History indicates that this form was brought into Greece, the threshhold of the Occident, from Egyptian, Indian, Babylonian and other Mid-Eastern and Eastern cultures.[1] In Greece during the 2½ to 3 centuries which separate Pythagoras from Euclid, a decisively different mentality began expressing itself through the same basic numerical and geometric structure which underlay the spiritually oriented cultures of the East. It is with Euclid that the ancient integration of the mystical combined with precise geometric and numerical speculation appears completely dissolved. Both Theon and Nichomachus provide us with an admixture of these two distinctly different techniques of thought, that is to say the more Eclectic, Aristotelian and Stoic in-

[1] A provacotive new study by Professor Ernest McClain, *The Myth of Invariance,* (New York, Nicholas Hayes, 1976) and his soon to be published study of the musical cosmologies of Plato add important insights into this transmission from Eastern cultures through Greece.

fluence was in their work brought into contact with the older Pythagorean and Oriental tradition. Although Theon and Nichomachus treat some of the same material found in the *Elements* of Euclid and must have been familiar with this work, still it is clear that they do not use Euclid in any principal way as a model. The approach is of an entirely different character. Euclid gives propositions with logical definitions and proofs and little or no philosophical penetration, while Theon and Nichomachus go back to the older Pythagorean model of announcing given Principles or *a priori* established laws accompanied with simple demonstrations for verification and contemplation, but always using these as a means for philosophizing about the nature of the Universe. Nevertheless the argumentative logistics so typical of the Greek mentality has made its inroads with these Neo-Pythagoreans at the same time as they attempt to preserve the ancient guidelines which make Mathematics a discipline for achieving spiritual knowledge.

Approached in this way, the entirety of Greek history can be interpreted as a drama of the struggle, conflict but ultimate emergence and domination of the rationalistic mentality in man's scientific endeavors. The Pythagorean position attempts to retain certain very specific ancient directives and warns against this movement which leads only toward a total empiricism. There is with them an implied axiom which seems to say that if man is to utilize successfully this faculty and power of observing and reasoning on existence then there must be imposed certain protective limitations on the procedures of the objectivizing intellect so that the fundamental spiritual Oneness is not obliterated in man's active consciousness. Throughout his exposition, Theon evokes or gives these directives and limitations. We may point out just a few: at the beginning of almost every exposition, Theon will say repeatedly "Starting from Unity," as if each engagement of the mind is to poise itself in the immutable oneness before it enters into the specificities, determinations and progressions which unfold out of the supra-rational Unit. The number One (or Unity) considered as the Mystical Monad is much more than the quantified unit: it is comparable to God and is the seed or seminal essence of everything which exists. He says,

> "Unity is the principle of all things and the most dominant of all that is: all things emanate from it and it emanates from nothing. It is indivisible and it is everything in power. It is immutable and

> never departs from its own nature through multiplication (1 x 1 = 1). All that is intelligible and cannot be engendered exists in it: the nature of ideas, God himself, the soul, the beautiful and the good, and every intelligible essence, such as beauty itself, justice itself, equality itself, for we conceive of each of these things as being one and as existing in itself."

This structure of a multiplicity that is counted out of [2] an omnipotent and uncognizable Unity is reiterated and imposed on all the patterns of thought made in Theon's thought throughout his exposition. The separation of this Unity into the fixed and volatile, that is the existence of a fixed or immutable ideal world of form "the eternally present thoughts of God"[3] which generate and sustain the vast progressions and permutations of this ever-changing transitory and evolving real world is a philosophic tone implicit in all Theon's numerical operations. Another teleological implication can be derived from the emphasis on an ecstatic reverence for the decad or tetractys (the number 10) among Pythagoreans. Again there is the implication that the universal outpouring of the monad into the tangibility of number is of a *cyclic* nature which ultimately results in a return of the all-containing synthesis of the original Unity, but through its transformation into the formal, embodied expression of this Unity in the form of the decad.

Only whole numbers or simple fractions are submitted to the reasoning exercises of Theon—numbers which through their arrangements in progressions relate to geometric formations, or numbers which in comparison with each other are related to musical ratios which produce vibratory tones in the human voice. The numbers in this way are connected to manifest, measurable realities. On the other hand these expositions exclude irrational numbers such as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ which arise in the perfection of pure geometric forms. These irrationals to the Pythagoreans came to represent the existence of an immeasurable, supra-rational world which could only be approached through heightened intellectual experiences above those of the rational mind, a boundary beyond which external knowledge has no place. The presence of the irrational in both geometric form and in musical sub-structure provided the indication to the ancient scientist of the necessity to move in higher and deeper levels of intelligence. This interesting interpretation of the Pythagorean viewpoint on the irrational and

[2] The Greek root of the word arithmetic is *arithmos* from verb *arithmein* = to count.

[3] The *Theologumena Arithmeticae.*

the change which occurred in the Greek mind is covered in detail in the chapter entitled "La Deviation" in *Le Roi de la Theocratie Pharaonique* and in other works by R.A. Schwaller de Lubicz who used Theon's work extensively in developing his theory of the relationship between Pythagorean teaching and the Temple knowledge of ancient Egypt.

We see, of course, in modern mathematics that these and other disciplines which help confine the rationalistic mental power to its proper domain have been completely lost as well as the acknowledgment that man's intelligence exists in a hierarchy of interrelated but qualitatively different mental processes to which our expression and modes of symbolization should conform. For the Pythagoreans could not regard numbers in the cold, impartial way of modern mathematics and Theon's writing demonstrates this throughout. He says: "Numbers are the sources of Form and Energy in the world, they are dynamic and active even among themselves. Hence, they convey in relationship to one another, specific qualities which take on an almost human characteristic in their capacity for mutual influence."[4]

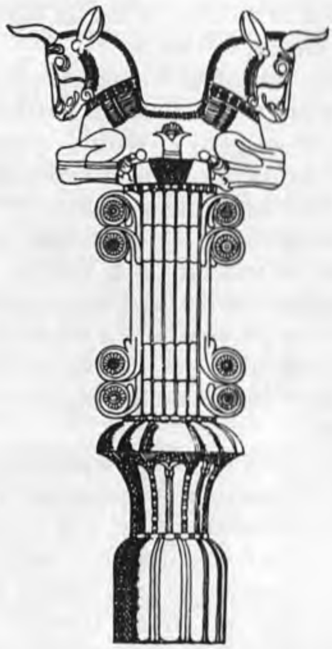
This living vitality attributed to numbers as active principles of Nature or as vibratory energetic determinations, leads us back in history tof the *Neters* or gods of Nature, while subsequent to the Greek era we find this tradition still of profound importance in the work of the great scientist Johannes Kepler (17th century), whose theory that the variations in the movement of the planets in our solar system could be transcribed as varying sonar patterns and rendered into a musical composition, has recently been verified and accomplished at Yale University by Professors Ruff and Rodgers using computers to calculate the relative velocity of the planets in their orbits and electronic sound devices to transform these computations into music.

Theon's work then is an important transmission of a doctrine incalculably old which appears and disappears in a variety of expressions through man's cultural history. It is a doctrine which, when approached with greater empathy, can release us from the rationalistic bias which caused our 19th century scholars to label the Pythagorean influences in Greek thought as "awkward" or "quaint" and "nonsensical." We can see that a system of numerical speculation which provides an unbroken connectiveness between

[4] see R.A. Schwaller de Lubicz, *The Temple in Man*, Boston, Autumn Press, 1977.

the things of mind, of body and of the ethical and spiritual nature, are not antiquated religious misconceptions, but speak of a philosophical and humanistic refinement. In turning our attention to these writers with a fresh attitude we see that Greece offers us a vantage point on the moment in our development when the misuse and distortions of the powers inherent in exact knowledge began, and of the ardent few, such as Theon, who tried to sustain a teaching which might have helped to prevent these distortions.

— *Robert Lawlor,* Tasmania, 1977



Earlier editions of Theon.

The only existing editions of Theon's work to this date (1892) are the following manuscripts in Latin:

1) *Theon of Smyrna the Platonist; Exposition of the works on Mathematics necessary in the understanding of Plato.* Work not published before, Latin version in Thuana (?) Library, with Illustrating Notes by Ismael Boulliau, Paris 1644. Greek-Latin Edition in quarto, of x, 308 pages, containing Arithmetic and Music. [Ismael Boulliau, 1605-1694]

2) *Inaugural Academic Specimen, Exhibiting the Arithmetic of Theon Smyrna, Boulliau version, Diverse lectures and annotations, Submitted for public examination by John Jacob de Gelder, 1827,* Greek-Latin edition of ixx, 200 pages printed in 8ᵉ; containing Arithmetic only.

3) *Book of Astronomy by Theon of Smyrna, First edition of text in the Latin version, with Geometric descriptions, dissertations and notes;* illustrated, by Th. H. Martin of the Literary Faculty in the Academia Rhedonensi, Paris 1849, Greek-Latin edition in 8ᵉ; containing Astronomy.

4) *Mathematic exposition as utilized by Plato, according to legend.* Reconstructed by Edward Hiller, Leipzig 1878, Greek Edition in 12ᵉ; of viii, 216 pages.

The first manuscript on parchment is of the 11th or 12th century. The second is on paper, in large format, of the 14th or 15th century.

The National Library in Paris is in possession of various manuscripts by Theon which have been numbered as follows: 1806, 1817, 1820, 2013, 2014, 2428, 2450 and 2460. This last ms. contains all of Theon's work on Music under the title: *Summary and outline on the whole of Music by Theon the Platonist.* In addition to the two manuscripts mentioned above which presently (1892) are in Venice, (apparently most of Theon's manuscripts are kept in the Saint Marc Library in Venice as well as the National Library in

Paris). M. Hiller also describes them in his preface of his Latin edition in the Saint Marc Library. The following manuscripts are known to be in the following libraries: ms. #512 and ms. #41 of the 13th or 14th century in the Ricardi Library in Florence; ms. #260 of the 15th or 16th century in the National Library in Naples; ms. #86 of the 16th century in the Berberini Library in Rome; and finally, in the Urbini Collection of the Vatican mss. #221 and #77, both of the 15th century.

The following libraries also possess manuscripts by Theon of Smyrna:
In England, the Trinity College of Cambridge University and the Bodleian Library of Oxford University.
In Spain, The Escorial Palace.
In Holland, in Leyden
In Italy, the Laurentian Library in Florence, the Ambrosian Library in Milan and the Royal Library in Turin.

Numerous passages have been collected from the manuscripts in the National Library in Paris and other various libraries in Italy.

J. Dupuis, 1892

The Saint Marc Library is also known as the *Marciaur*. The Ricardi Library as Laurenziand. The above translation is of Dupuis introduction to the 1892 Greek/French edition, and in the intervening years the colloquial references have doubtles changed.

THEON OF SMYRNA

THE MATHEMATICAL KNOWLEDGE USEFUL FOR THE READING OF PLATO

PART ONE

On the Usefulness of Mathematics

INTRODUCTION

I. Everyone assuredly will agree that it is not possible to understand what Plato has written on mathematics if one is not devoted to their study. He himself has shown in many places that this knowledge is not useless and without results in the other sciences. Therefore the person should be considered fortunate who, in approaching the writings of Plato, has a thorough knowledge of geometry, music and astronomy. But these are areas of knowledge whose acquisition is neither rapid nor easy, on the contrary they require assiduous work from early youth. In the fear that those who have not had the opportunity to cultivate mathematics, and who nevertheless desire to know the writings of Plato, might be forced to renounce this desire, we will give here a summary and abridgment of the necessary knowledge and the most useful of the traditional mathematical theorems on arithmetic, music, geometry, stereometry and astronomy, sciences without which, as Plato said, it is impossible to be completely happy.[1]

Eratosthenes, in the book entitled *The Platonist,* reports that the Delians, having consulted the oracle as to how to save themselves from the plague, were perscribed by the God to construct an altar

[1] Plato, *Epinomis,* Collected Dialogues, edited by E. Hamilton and H. Cairns, Bollingen Series LXXI, Princeton University Press p. 992a.

double the size of the one which already existed. This problem threw the architects into a strange embarassment. They wondered how one could make one solid the double of another. They questioned Plato about this difficulty. He answered that the God had thus sent this divination not because he had any need of a double altar, but in order to reproach the Greeks for neglecting the study of mathematics and for belittling the value of geometry.[2]

In order to enter into these views of Pythian Apollo, he expanded at length in his conversations on the usefulness of mathematics. Thus, in the *Epinomis,* wishing to arouse the student, he said, "Certainly no one in the State would be happy if they are ignorant of them; such is the way, such is education, such are the sciences, easy or not to learn, which can lead to this goal; one has not the right to neglect the Gods..."[3] Further on he says again: that "if there is among you one who is such (mathematician), it is he who will be favored with good fortune and overflow with wisdom and felicity".[4]

In *The Republic* he writes, "Starting at twenty-five years, those who are chosen will achieve most honorable distinctions and they must be presented with a totality of the sciences, which every one of them, in childhood, had studied separately, so that they will grasp a general point of view, and the relationships that these sciences have to one another, and to the nature of being."[5]

He prescribes that one devote oneself first to the study of arithmetic, then of geometry and in the third place to stereometry, and following that to astronomy, which he said to be the study of the solids in movement. Finally, he urged in the fifth place the learning of music. After having shown the usefulness of mathematics, he said, "You are amusing, you who seem worried that I impose impractical studies upon you. It does not only reside with mediocre minds, but all men have difficulty in pursuading themselves that it is through these studies, as if with instruments, that one purifies the eye of the soul, and that one causes a new fire to burn in this organ which was obscured and as though extinguished by the shadows of the other sciences, an organ whose conservation is more important than ten thousand eyes, since it is

[2] See note 1 after the translation.

[3] *Epinomis,* passage cited.

[4] *Epinomis,* 992b

[5] *Republic,* VII 537b. Plato's text says 20 instead of 25 years.

by that alone that we contemplate the truth."[6]

In the seventh book of the *Republic,* speaking of arithmetic, he says that it is the most necessary study, since all the arts, all the conceptions of our minds, all the sciences and even the military art depend upon it. "Palamedes," he says, "in the tragedies, often represents Agamemnon as a ridiculous general who boasted of having invented the numbers and of having put the camp and the fleets of the Greeks before Ilion and all the rest in order whereas previously no numbering was made, and that Agamemnon himself seemed not to have known how many feet he had, for he was completely ignorant of the art of counting. Arithmetic seems therefore by its nature to belong to that which raises the soul to pure intelligence, and leads it to the contemplation of being, but no one makes the correct use of it."[7] Things that make only a single impression on the senses do not at all invite the understanding to reflection: such is the viewing of one finger, be it thick or thin, long or short, but that which gives birth to two opposite sensations has the power to awaken and excite our understanding, as when the same object appears large or small, light or heavy, one or multiple. It is therefore Unity and the number which have the virtue to awaken and excite our intelligence, since that which is *one*† sometimes appears multiple. The science of calculation and arithmetic leads us then to the knowledge of truth.[8] "The art of calculation must therefore not be treated in the manner of the populace, but in a way which directs men to contemplation of the *essence* of numbers, not from the viewpoint of commerce as with merchants and peddlers, but for the good of the soul in facilitating for it the means of elevating the order of transitory things towards the truth of being. It is indeed this study which, giving our soul a powerful impulse towards the upper region, obliges it to reason on the numbers as they are in themselves, without ever allowing the discussion to rest upon the visible and tangible objects."[9] He says again in the

[6] *Republic* VII, 537d, the text of this quotation and the following differ noticeably from Plato's. Plutarch seems to have partially copied the passage when he said: "Accustomed, by strong attachments to suffering and to pleasure, to taking the incertain and changing substance of bodies for real being, the intelligence becomes blind to the regard of true being: it looses the organ which by itself alone is worth ten thousand eyes, I mean the vision of the soul's light through which alone divinity can be seen." *Symposiacs,* VIII, quest. II, 1, p. 718e.

[7] *Republic* VII 522-523d, Poets have borrowed this language of Palamedes in several tragedies where they have him play a role.

[8] Cf. *Republic,* VII 525b. *Philebus,* pp. 14 and the following. *Gorgias,* pp. 450 d-451 c. Notice the distinction that Plato established between the science of calculation and the science of numbers.

[9] Cf. *Republic* VII, 525cd.

† *The One,* Ἓν = one (as number) or unit = Μονάς (monad.) The one and the number - the number being the expression or manifestation of the *unit,* or one) (Toulis)

same book, "Those who know how to calculate apply themselves successfully to all the sciences and even those who have a slower mind become more intelligent through this."[10] In the same book he again assures us that even in war, the art of calculation is very useful, "for encampments, for taking possession of areas, for the concentration and deployment of 'troops".[11] Further on, praising the same science he says that geometry pertains to surfaces, but "that astronomy has as its object the movement of the solid, which consequently obliges the soul to look upward and to bypass the things of the earth for the contemplation of those of the sky."[12] In the same writing he speaks of music because for the contemplation of all that exists, two things are necessary....."Astronomy and Harmony which, according to the Pythagorean doctrine, are two sister sciences."[13] Those therefore, strive in vain who seek to know diatonic nuances and to compare sounds by being satisfied by attentively straining their ear and drawing as near as possible to the instrument, as if wishing to overhear secretly the conversation of their neighbor.[14] Some say that they hear a certain particular sound between two sounds and that the interval is the smallest that can be detected. Others doubt the existence of this sound. Preferring the authority of the ear to that of the spirit, they seek the truth in the plucking of the strings and the turning of the keys of their instruments. But the clever arithmetician seeks in reflection for what numbers correspond to the consonances and form harmony, and what ones are those which correspond to dissonances.[15] This study leads to the investigation of the good and beautiful, all else is useless. Any method, if it is general and extends to all the common properties of things through drawing tighter the bonds of their mutual affinities, will bear fruit according to the ardour and zeal with which one applies one's self to it. It is indeed impossible that the dialecticians who are so clever, do not know how to account for the reason of things to themselves nor to others.[16] No one will arrive at this if he does not take these sciences for guide, for it is in reasoning according to them that we arrive at the contemplation of things.

[10] Cf. *Republic* VII, 526b.
[11] *Republic* VII 526d.
[12] *Republic* VII, 529a.
[13] *Republic* VII, 530d.
[14] *Republic* VII, 531a; see further on, section II.
[15] *Republic* VII, 531c.
[16] *Republic* VII, 531d.

In the *Epinomis,* Plato comes again to Arithmetic which he calls a "gift of God"[17] and he says that no one would know how to become virtuous without it. Then passing to the description of the contrary, he says, "If one were to remove numbers from humanity, all discretion would be made impossible: the soul of the animal deprived of reason would be incapable of any virtue; moreover it would no longer have its essence. Certainly the animal, who knows how to distinguish neither two nor three, who knows neither the even nor the odd, who knows nothing of number, will never be able to give reason to anything, knowing only through the senses and the memory. Deprived of true reason he will never become wise." [18] Let us pass in review everything which is related to the other arts, and we will see that there is nothing which would subsist, nothing which would not perish were the science of number withdrawn. To consider only the arts, one would be reasonably able to believe that this science is only necessary to the human species for objects of little importance; this would already be much to concede. But he who will consider that there is something divine in the origin and the mortality of man, sees what need he has of piety towards the gods, and that person will recognize number in man, and unless he be a prophet, he will never know nor comprehend for how many of our faculties and forces number is the source. It is evident, for example, that music can only result from movements and sounds measured by numbers, and it is not less evident that number, as the source of all that is good, could not be the cause of any "evil". On the contrary, that which is devoid of number lacks any sort of reason; it is without order, without beauty, without grace and ultimately deprived of all perfections. Further on, he continues thus: "No one will ever pursuade us that there could be a virtue for the human species greater or more noble than piety"[19] because it is through piety that he who has taken care to instruct himself acquires the other virtues. He next shows how one inspires piety towards the gods; then he says that it is necessary to begin with astronomy, because if one is ashamed to commit falsehood in the eyes of men, one is even more ashamed to do so in the eyes of the gods. Now he is a mendacious person who develops a false opinion of the gods and expresses it without even having

[17]I believe", he says, "I may say that t'is not so much our luck as a god who preserves us by his gift of it," *Epinomis,* p. 976 e.

[18] *Epinomis,* p. 977d.

[19] *Epinomis,* 989b.

studied the nature of the perceptible gods, that is, astronomy. "Do you not know", he says, "that the person who is truly an astronomer is necessarily very wise, not an astronomer in the manner of Hesiod who busies himself with observing the rising and setting of the stars, but of those who scrutinize the revolutions of the seven planets, the knowledge of which all the genius of man finds it difficult to achieve."[20] Now he who proposes to prepare the minds of men for these studies, which suppose so much preliminary knowledge, must have himself been given familiarity with the mathematical sciences since childhood and all during his youth. And among these sciences, the best, the principal one is the science of abstract numbers, separate from all matter, also that of generation and the properties of the odd and even, as far as they contribute to making known the nature of things.[21] There is an aspect of this science which has been given the completely ridiculous name of geometry, because it contains a combination of numbers which are not similar to one another by nature, a combination which makes evident the condition of surfaces. Following this he mentions another experience and art which he calls stereometry: if someone, he says, will multiply three numbers whose planes are as I already discussed, making the product similar (to another number) and dissimilar to what it was, he will create a solid body *Στερεά Ποιεῖ Σῶματα* which is truly a divine and marvelous work.[22]

In the *Laws,* speaking of musical harmony, he says that, "the greatest and beautiful political harmony is wisdom. One possesses it only to the extent that one lives according to right reason; as for the one who abandons it, he is the corruptor of his own house. He is a citizen useless to the safety and prosperity of the State, since he lives in extreme ignorance."[23]

And in the third book of the *Republic,* wishing to prove that the philosopher alone is a musician, he says:

> "By the immortal gods, we will never be musicians, neither we nor the guardians whom we must educate, as long as we do not know all the forms of temperance, of courage, of generosity and of high mindedness and everything in the world which conforms to or is contrary to these virtues, and how to recognize them or to recog-

[20] *Epinomis,* p. 990a.

[21] *Epinomis, p.990c.*

[22] *Plato is undoubtedly alluding to this problem: "construct a rectangular parallelopiped similar to a given rectangular parallelopiped which is to this solid in a given relationship", a problem of which that of the duplication of the cube is but a particular case.*

[23] *Laws,* III, p. 689d.

> nize their images in those who possess them, without neglecting any of them, large or small, regarding them as being part of the same art and the same study."[24]

By these words and by the preceeding he proves the usefulness of music, and shows that only the philosopher is truly a musician, while he who is vicious and mean is a stranger to the Muses. For, he says, the true and sincere integrity of morals, this virtue which consists in the good and honest ruling of our life, follows right reason, that is to say, usage conforms to reason. He adds that the companions of right reason are decency, cadence and accord; decency in song, accord in harmony, cadence in rhythm. On the other hand, impropriety or moral corruption is essentially linked to a perversion of reason, or to the corrupted use of reason, and its companions are indecency, confusion and discord in all that one does by oneself or by imitation, so that only he who has good morals is a musician, and as can be seen by the preceding, he is also the true philosopher, if, always since the first years of his adolescence, when he was taught music, he took on habits of decency and order, for music joins innocent pleasure to utility. It is impossible, says Plato, for someone to become a perfect musician who has not acquired all the habitudes of a good education, who does not have ideas of decency, of nobility of the soul and of temperance. He must realize that these ideas are found everywhere and are not to be taken lightly either in large things or in the small. Because it belongs to the philosopher to know ideas, no person will know modesty, temperance and decency if he himself is immodest and intemperant. But the things which comprise the embellishment of human life, beauty, harmony, honesty, all these are the images of that beauty, that accord, that beautiful eternal order which has a true existence, that is to say that these perceptible things are the characteristics and the expression of intelligible things, or ideas.

The Pythagoreans, whose feelings Plato often adopted, also define music as the perfect union of contrary things, unity within multiplicity, even accord within discord. For music does not only coordinate rhythm and modulation, it puts order into the whole system; its end is to unite and to coordinate, and God also is the arranger of discordant things, and his greatest work is to reconcile with one another things which were enemies, by means of the laws of music and medicine. It is also through music that the harmony of things and the government of the universe is maintained, for that

[24] *Republic*, III, p.402b.

which harmony is in the world, good legislation is in the State, and temperance is in the family. It has indeed the power to bring order and union into the multitude. Now the efficacy and use of this science, says Plato, is seen in four of the elements belonging to humanity: mind, body, family, and State. Indeed, these four things have need of being well ordered and constituted.

Here again is what Plato says of mathematics in the books of the *Republic:*

> "The good man is he who, tried by pain or pleasure, agitated by desire or by fear, always preserves, without ever rejecting them, the right ideas which have been given to him in his education. I am going to tell you of something which appears to me to be similar. When our dyers want to dye wool purple, they begin by choosing from among the wools of various colors, that which is white. Then they make their preparation, and no little care is necessary so that the wool takes the color in the best way. This is how they proceed, and thanks to this method, the colors are incorporated into the wool, and their brilliancy cannot be removed, neither with the aid of lye nor otherwise. But if, on the contrary, the dyer does not take these precautions, one knows what happens and how the wools retain little of their color, which comes off and disappears. It is necessary to proceed in the same way with our faculties." [25]

We teach children music, gymnastics, letters, geometry and arithmetic, neglecting nothing in order that they receive, like a dye, the reasons for all the virtues we teach them; after having previously administered to them detergents and other preparations, consisting in these sciences which are as so much astringent medicine, their feelings will remain indelible, their character will have been formed by education. This color and this dye that we will have given them will not be removable by any lye—by this I mean sensual pleasure, more dangerous than any perversity and any habitude,—nor by pain nor by fear and greed, more corrosive than any lye.

We can again compare philosophy to the initiation into things truly holy, and to the revelation of the authentic mysteries.[26] There are five parts in initation: the first is the preliminary purification, because participation in the mysteries must not be indiscriminately given to all those who desire it, but there are some aspirants whom the harbinger of the path separates out, such as those of impure

[25] *Republic*, IV, p.429d.
[26] Cf. *Phaedo*, p.69d.

hands, or whose speech lacks prudence; but even those who are not rejected must be subjected to certain purifications. After this purification comes the tradition of sacred things (which is initiation proper). In the third place comes the ceremony which is called the full vision (the highest degree of the initiation). The fourth stage, which is the end and the goal of the full vision, is the binding of the head and the placement of the crowns, in order that he who has received the sacred things, becomes capable in his turn of transmitting the tradition to others, either through the *dadouchos* (the torch bearing ceremonies), or through hierophantism (interpretation of sacred things), or by some other priestly work. Finally the fifth stage, which is the crowning of all that has preceeded it, is to be a friend of the Diety, and to enjoy the felicity which consists of living in a familiar commerce with him.

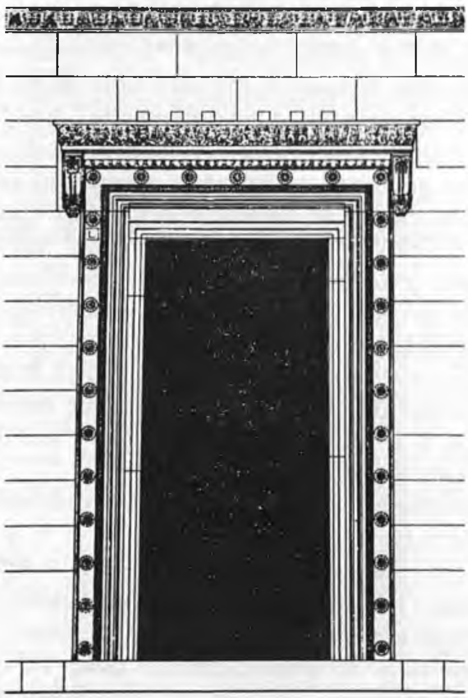
It is in absolutely the same manner that the tradition of Platonic reason follows. Indeed one begins from childhood with a certain consistent purification in the study of appropriate mathematical theories. According to Empedocles,[27] "it is necessary that he who wishes to submerge himself in the pure wave of the five fountains begins by purifying himself of his defilements." And Plato also said one must seek purification in the five mathematical sciences, which are arithmetic, geometry, stereometry, music and astronomy. The tradition of philosophical, logical, political and natural principles corresponds to initiation. He calls full vision[28] the occupation of the spirit with intelligible things, with true existence and with ideas. Finally, he says that the binding and the crowning of the head must be understood as the faculty which is given to the adept by those who have taught him, to lead others to the same contemplation. The fifth stage is that consummate felicity which they begin to enjoy, and which, according to Plato, "identifies them with the Diety, in so far as that is possible."

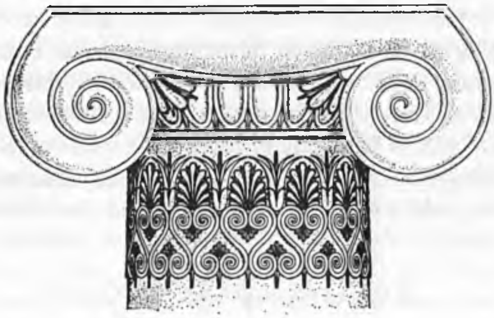
One who wishes to demonstrate the usefulness and the necessity of the mathematical sciences could speak of it at even greater length than this. But from the fear that I might appear unable to stop presenting the reasons to praise these sciences, I am going to begin the explanation of the necessary theorems, all of which would be necessary for the readers in order to become perfect arithmeticians, geometers, musicians or astronomers, but that is not

[27] *Empedocles,* vs. 432, edition Mullach.

[28] Cf. *Phaedrus,* p. 250c.

the goal sought by all those who wish to read the writings of Plato; here I will explain the theorems which are sufficient for understanding the meaning of his writings. Indeed Plato himself did not wish that we continue to extreme old age to draw geometric figures or to sing songs, things which are appropriate for children and which are destined to prepare and purify their minds in order to make them capable of understanding philosophy. It is sufficient that those who wish to approach our writings or those of Plato might have gone through the first elements of geometry, so that they may readily understand our explanations. However, what we say will be such that it can be understood even by those who are completely ignorant of mathematics.





ARITHMETIC

ON THE ORDER IN WHICH MATHEMATICS MUST BE STUDIED

II. We are going to begin with the arithmetic theorems which are very closely connected with the musical theorems which are transposed into numbers. We have no need for a musical instrument, as Plato himself explains, when he says that it is not necessary to agitate the strings of an instrument (with hand to ear) like curious folk trying to overhear something. What we desire is to understand harmony and the celestial music; we can only examine this harmony after having studied the numerical laws of sounds. When Plato says that music occupies the fifth rung [29] (in the study of mathematics), he speaks of celestial music which results from the movement, the order and the concert of the stars which travel in space. But we must give mathematics of music second place (that is to say, place it) after arithmetic, as Plato wished, since one can understand nothing of celestial music, if one does not understand that which has its foundation in numbers and in reason. Then so that the numerical principles of music can be connected to the theory of abstract numbers, we will give them the second rung, in order to facilitate our study.

According to the natural order, the first science will be that of numbers, which is called arithmetic. The second is that whose object is surfaces and is called geometry. The third, called stereome-

[29] Plato places music after astronomy — *Republic* VII, 530d — after having assigned the fourth rung to astronomy — *Republic* VII, 528e.

try, is the study of solid objects. The fourth treats of solids in movement and this is astronomy. As for music, whose object is to consider the mutual relations of the movements and intervals, whatever these relations be, it is not possible to understand it before having grasped what is based on numbers. Thus in our plan, the numerical laws of music will come immediately after arithmetic; but following the natural order, this music which consists in studying the harmony of the worlds will come in fifth place. Now, according to the doctrine of the Pythagoreans, the numbers are, so to say, the principle, the source and the root of all things.

THE ONE AND THE MONAD

III. Number is a collection of monads, or a progression of the multitude beginning from and returning to the monad (through the successive addition or subtraction of one unit). As for the monad, it is the terminate quantity—the principle and element of numbers, which, when disentangled from the multitude through subtraction and isolated from all number, remains firm and fixed: it is impossible to push division any further. If we divide a tangible body in several parts, what was *one* becomes several, and if each part is then subtracted, it will terminate at *one;* and if we again divide this *one* into several parts, the multitude will arise out of it, and in taking away each of these parts, we come again to *one* in such a way that that which is one, as one, is without parts and indivisible.[30] Every other number, being divided, is diminished and reduced into parts smaller than itself, like six into 3 and 3, or into 4 and 2, or 5 and 1. Among tangible things, if that which is one is divided it is diminished in bodily size and as a result of the partitioning it is divided into parts smaller than itself, yet it grows larger as a number because in place of that which was one there are now several. And it follows that this one is indivisible. Indeed nothing can be divided into parts larger than itself. But that which is one, divided into parts greater than its whole, is also divided, in the manner of numbers, into parts equal (in sum) to the whole. For example, if a body, a tangible unity, is divided into six parts, 1, 1, 1, 1, 1, 1, these parts are equal to the unit; but if it is divided into 4 and 2, the parts are greater than the unit, indeed 4 and 2 as numbers surpass one. The monad, then, as a number, is indivisible. But it is called the

[30] See note II after the translation.

monad because it remains immutable and does not go beyond the limits of its nature, indeed in multiplying the monad by itself we will always have the monad; one times one is always one; and if we multiply even to infinity it will always remain the monad. Moreover, it is called the monad because it is separated and placed alone outside the multitude of other numbers. As number differs from that which is numbered, in the same way the monad differs from that which is one. The number indeed is an intelligible quantity, like the quantity 5 and the quantity 10 which are not composed of tangible bodies, but of intelligible ones. However, the numerable quantity is found among tangible things, such as 5 horses, 5 oxen, 5 men. As for the one which is met among tangible things, it is called *one* in itself, as one horse, one man.[31]

IV. The monad is then the principle of numbers; and the *one* the principle of numbered things. That which is one, being tangible, can most assuredly be divided to infinity, not so far as it is number or the principle of number, but in so far as it is tangible, so that the monad which is intelligible does not admit of division, but that which is one, being tangible, can be divided to infinity. Numbered things again differ from numbers in that they are corporeal, while the numbers are incorporeal. But, with a naive attitude, posterity considers the monad and the dyad as principles of numbers; whereas for the Pythagoreans, the principle of numbers consists in the series of successive terms through which the odd and even are conceived." They say, for example, that the principle of three among tangible things is the triad, and the principle of all that which is four, among tangible things, is the tetrad, and similarly for all other numbers. They further claim that the monad is the principle of all these numbers and that the *one* is free of all variety, the one which is found in numbers is not just any one, that is to say, it is not a certain quantity and a diversity with regard to another one, but is the one considered in itself. For it is through this that it becomes the principle and the measure of things which are subordinate to it, in the same way that each existing thing is called one, as being a participant in the first essence and idea of that which is one. Archytas and Philolaus use the words one and monad interchangebly, and they say that the monad is the one. Most add the epithet "first" to the name monad, as if they had a monad which was not first, and as if the one which they call first were the more universal,

[31] Thus according to Theon, the monad is abstract, the *one* concrete.

and were both the monad and the *one* — for they also call it the one — and as if it were the first and intelligible essence which caused all the things that are one to be such. It is by virtue of a participation in this essence that all things are called one. This is why the name itself, *one,* does not specify what the object is nor of what species it partakes, but is applied to all things. Thus the monad and the *one,* being at the same time intelligible and tangible, are in no way different from one another. Others place another difference between the *one* and the monad: the *one* does not change according to the substance, and it is not the *one* which causes the monad or the odd numbers to change according to essence. Neither does it change according to quality, for it itself is monad, and unlike the monads which are several. It does not change according to quantity either, because it is not composite, as are the monads to which can be added another monad. It is one and not several; it is for this reason that it is called the unique *one.* And although Plato, in the Philebus[32] uses the expression "the Units" [33], he did not call them so after one, but after the monad which is a participation in the one. This one which is distinguished from the monad of which it is the essence, is something completely immutable. The *one* then differs from the monad in that it is defined and terminated, while the monads are indefinite and indeterminate.

ODD AND EVEN NUMBERS

V. Initially the numbers are divided into two kinds: those which are called even and the others which are called odd. The even numbers are those which can be divided into two equal parts, such as two and four; the odd numbers, on the contrary, are those which can only be divided into unequal parts, like five and seven. Some have said that the first of the odd numbers is the monad (unit) for even is the opposite of odd, and the monad (unit) is necessarily either even or odd. Now it cannot be even since, not only does it not divide into two equal parts, but it does not even divide at all; therefore unity, is odd. If you add an even number to another even number, the total will be even; now unity added to an even number gives an odd number; therefore again, unity is not even, but odd. However, Aristotle says, in *The Pythagorean*[34] that *one* participates

[32] Philebus, p. 15a.

[33] A variation of the word Μονας = unit or monad here referred to as an adj. Ενας = unit (From Ενας = one)

[34] One of the lost works of Aristotle.

in both natures. Indeed, added to an odd number it gives an even number, which it would not be able to do if it did not participate in both natures. This is why it is called *odd-even.* Archytas also appears to have had this feeling. The first idea of the odd is therefore the monad, as also in the world, the quality of odd is attributed to that which is defined and well ordered. On the other hand, the first idea of the even is the indefinite dyad[35], which causes that which is indefinite, unknown and disorderly to be attributed to the quality of even, in the world also. This is why the dyad is called indefinite, because it is not defined, as is the Unit (or monad). As for the terms which follow in a continued series, beginning with unity, they always increase by an equal quantity, each surpassing the preceeding one by a unit. But in the measure of the terms, 1, 2, 3, 4, 5, 6, the ratio of the number 2 to unity is double; that of 3 to 2 is sesquialter ($1 + \frac{1}{2}$) finally that of 6 to 5 is sesquiquintan ($1 + \frac{1}{5}$). Now the relationship $1 + \frac{1}{5}$ is smaller than $1 + \frac{1}{4}$; $1 + \frac{1}{4}$ is smaller than $1 + \frac{1}{3}$; $1 + 1\frac{1}{3}$ is smaller than $1 + \frac{1}{2}$; and finally $1 + \frac{1}{2}$ is smaller than 2. And one would find that the ratio decreases in like manner for the other numbers. It can thus be seen that the successive numbers are alternately odd and even.

PRIME OR INCOMPOSITE NUMBERS

VI. Some among the numbers are called absolute prime or incomposite numbers; others are prime in relation to each other but not absolute; others are absolutely composite and still others are composite in relation to each other. The absolute prime numbers (incomposite) are those which no other number, except the unit, can measure. These are 3, 5, 7, 11, 13, 17...and other similar numbers. These numbers are also called linear and euthymetric, because the lengths and the lines are only considered as a single dimension. They are also called the oddly-odd numbers. They have thus been given five different names: prime, incomposite, linear, euthymetric and oddly-odd. They are the only indivisible numbers; thus none of the numbers other than unity (monad) can divide 3 in such a way that 3 could result from their multiplication. Indeed one times 3 is 3. Likewise, one times 5 is 5, one times 7 is 7, and one times 11 is 11. And for this reason these numbers are called oddly-

[35] Dyad = two/pair from Greek δυάς, Triad = three/trinity from Greek τριάς (ἅγια τριάς = holy trinity) (Toulis)

odd [36] for they are odd and unity which measures them is odd as well. Also, only odd numbers can be prime and incomposite. Indeed the even numbers are not prime or incomposite; it is not only unity which measures them, but other numbers also. For example, the dyad measures 4 because 2 times 2 makes 4; 2 and 3 measure 6 because 2 times 3 and 3 times 2 make 6. All the other even numbers with the exception of 2 are likewise measured by numbers greater than the unit. The number 2 is the only one among the even numbers which is similar to the odd numbers in having only unity for its measure. Indeed one times two is two. Because of this it is said that the number two has the nature of the odd numbers because it has the same property as the odd. Those which are called prime in relation to one another, but not absolute, are those which have unity for common measure, although other numbers measure them if they are considered separately, such as 8 which is measured by 2 and 4, 9 which is measured by 3, and 10 which is measured by 2 and 5. They indeed have unity for common measure either in relation to one another or in relation to their prime factors: one times 3 equals 3; one times 8 equals 8, one times 9 equals 9, and one times 10 equals 10.

COMPOSITE NUMBERS

VII. Composite numbers are those measured by a number smaller than themselves, like 6 which is measured by 2 and 3. Composite numbers in relation to one another are those which have a common denominator, such as 8 and 6 which have 2 as their common denominator, for 2 times 3 equals 6 and 2 times 4 equals 8. Such again are 6 and 9 which have 3 for their common denominator, for 3 times 2 equals 6 and 3 times 3 equals 9. As for the unit, it is not a number, but the principle of number; and, as for the number 2, it is not indefinite,† it is the first number different from the unit, and although even, it does not have a divisor greater than the unit. The composite numbers which are the product of two numbers are called *planar,* they are considered as having two dimensions, length and width. Those which are the product of three numbers are called *solids* since they possess the third dimension. Finally the numbers resulting from the multiplication of one type of numbers by the other are called *peripheral* numbers.

[36] Euclid calls numbers in the form $(2a + 1)(2b + 1)$ oddly odd, cf. *Elements* VII, def. 10. Prime numbers are included in this formula by supposing that $2b + 1 = 1$, that is to say, $b = 0$.

† Unity + unity= the indefinite dyad, but $1 + 1 = 2$ is 'not indefinite.'

VARIOUS KINDS OF EVEN NUMBERS

VIII. Among the even numbers, some are evenly-even, and others are oddly-even, and still others are evenly-odd. It is to be observed that a number is evenly-even when it combines these three conditions: (1) that it be created by two even numbers multiplied by each other; (2) that all the parts be even down to the final unit; (3) that none of its parts is homonymous to the odd number. Such are 32, 64, 128, and those which follow in proceeding by a double progression. Indeed, 32 is the product of the numbers 4 and 8 which are even. All its parts are even; its half, 16; its quarter, 8; its eighth, 4. These parts are homonymous to the even numbers, the half is considered as the binary number, and the same for the quarter and the eighth (which are considered as the numbers 4 and 8). This ratio applies also for the other numbers.[37]

IX. The evenly-odd numbers are those measured by the number 2 and by any odd number, and which consequently have odd halves when divided in equal parts. Such, for example, is 2 times 7 or 14. They are called evenly-odd because they are measured by the dyad, which is even, and an odd number besides; 2 has the unit; six has the number 3; 10 has the number 5; 14 has 7. These numbers, when divided once by two, are partitioned into two odd parts, and after this first division in equal parts, they will not be divided equally any further. Indeed half of 6 is 3, but 3 cannot be divided into equal parts, for unity (which remains after the division by 2) is indivisible.[38]

X. The oddly-even are those which result from the multiplication of any two numbers, one of which is odd and the other even, which when multiplied, are divided by the number 2 into two even parts; but if a larger divisor is used, the quotients are sometimes even, sometimes odd. Such are the numbers 12 and 20, which respectively have the value 3 times 4, and 5 times 4. Now, in dividing 12 successively by 2, 3, and 4, we have $12 = 2 \times 6 = 3 \times 4 = 4 \times 3$. Likewise we have $20 = 2 \times 10 = 4 \times 5 = 5 \times 4$.[39]

[37]Thus, according to Theon, the evenly-even number is a power of 2. According to Euclid, it is a product of two even numbers; cf. *Elements*, VII, def. 8.

[38] The evenly-odd numbers are then, according to Theon, the numbers in the form $2(2a + 1)$. This is the same definition as that of Euclid, Cf. *Elements*, VII, def. 9.

[39] The oddly-even numbers, which Theon distinguishes from the evenly-odd, would then be numbers taking the form $(2a + 1) 4b$.

ON EQUALLY-EQUAL, UNEQUILATERAL AND[40] PARALLELOGRAMATIC NUMBERS

XI. Among composite numbers, some are equally-equal, that is square and planar, when they result from the multiplication of two equal numbers (the result is equally-equal, and square). Such are the numbers 4 and 9, for 2 times 2 is 4 and 3 times 3 is 9.

XII. On the contrary, composite numbers are unequally unequal when they result from the multiplication of two unequal numbers. Such is 6, since 2 times 3 is 6.

XIII. Among these numbers, those which have one side (factor) longer than the opposite by one unit are called unequilateral. Now, the number which surpasses the odd number by one unit is even, thus unequilaterals are only comprised of even numbers. Indeed, unity, the principle of all numbers, being odd and tending to the production of the others, makes, through doubling itself, the dyad, which is unequilateral. This is why the number 2, being unequilateral and surpassing unity by one, renders the even numbers which surpass the odd numbers unequilateral by one unit. Now numbers are created in two ways, by multiplication and by addition. The even numbers which are added to the even numbers which precede them produce unequilateral numbers. Thus these are the successive even numbers,

2, 4, 6, 8, 10, 12, 14, 16, 18.

Through addition, we have $2 + 4 = 6$; $6 + 6 = 12$; $12 + 8 = 20$; $20 + 10 = 30$; so that the sums are the unequilateral numbers 6, 12, 20, 30 and so forth.[41]

The same unequilateral numbers are also obtained by the multiplication of even and successive odd numbers, the first number being multiplied by the following:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

We have 1 times $2 = 2$; 2 times $3 = 6$; 3 times $4 = 12$; 4 times $5 =$

[40] Unequilateral - from Greek Ἑτερομήκης which means of different length (ἕτερος+ μῆκης) other length

[41] The sum of the terms of the progression formed by the natural series of even numbers

2, 4, 6, 8, 10, 12, 14, 16, 18...2N

is in effect $n(n + 1)$, therefore it is an unequilateral number by definition.
Theon never gives the demonstration of the arithmetic theorems which he states; he verifies them with several examples.

20; 5 times 6 = 30; and so forth. The unequilateral numbers are given this name because the addition of unity to one of the sides make the first diversity of the sides.

XIV. The parallelogrammatic numbers are those which have one side greater than the other side by 2 units or greater, like 2 times 4, 4 times 6, 6 times 8, 8 times 10, which yield 8, 24, 48, 80 respectively.

XV. The numbers created by the addition of the successive odd numbers are square. Thus indeed in the series of odd numbers, 1, 3, 5, 7, 9, 11, 1 and 3 make 4 which is square, for it is equally equal, 2 times 2 make 4; 4 and 5 make 9, which is also square, since 3 times 3 equals 9; 9 and 7 make 16, which is square because 4 times 4 = 16; 16 and 9 make 25, which is again a square number because it is equally equal, 5 times 5 make 25. One could continue this way to infinity. Such is therefore the creation of the square numbers by addition, each odd number being successively added to the square obtained by the summation of the preceeding odd numbers starting from unity [42] The creation also takes place through multiplication, by multiplying any number by itself, such as 2 times 2 equals 4; 3 times 3 equals 9, 4 times 4 equals 16.

XVI. The mean terms for the consecutive squares in geometric proportion are the unequilateral numbers, that is to say the numbers whose one side is longer than the other by one unit; but the consecutive unequilateral numbers do not have squares for their proportional means.

Thus, taking the numbers 1, 2, 3, 4, 5; each of them multiplied by itself gives a square: 1 x 1 = 1; 2 x 2 = 4; 3 x 3 = 9; 4 x 4 = 16; 5 x 5 = 25; none of the factors go beyond their own limits; for the number 2 can only double itself, the number 3 can only triple itself... The successive squares are thus 1, 4, 9, 16, 25. I assert that they have the heteromecis for a mean. Let us actually take the successive squares 1 and 4, the mean between them is the unequilateral number 2; if we take the series 1, 2, 4, the mean 2 contains the extreme 1 as many times as it is contained in the other extreme 4; 2 is, indeed, the double of 1, and 4 the double of 2. Taking again the squares 4 and 9, their mean is the unequilateral 6. If we put 4, 6 and 9 in a line, the relationship of the mean 6 to the first extreme is equal to the relationship of the second extreme to 6 because the relationship of 6 to 4 is sesquialter (1 + ½), like the

[42] In fact the nth odd number starting from the unit is $2n - 1$ and the sum of the terms of the progression, 1, 3, 5, 7, 9....$2n - 1$ is n^2.

relationship of 9 to 6. The same is so with the following squares.

The unequilaterals on the contrary, products of factors which differ by one unit, do not remain in their own limits and do not include the squares. Thus 2 x 3 = 6; 3 x 4 = 12; and 4 x 5 = 24. Now none of the (first) factors stay within their own limits, they change in the multiplication. The number 2 is multiplied by 3, the number 3 is multiplied by 4 and 4 by 5.

Furthermore, the created unequilateral numbers do not include the square numbers. Thus 2 and 6 are the successive unequilaterals between which is found the square 4, but 4 is not contained between them according to a continuous geometric progression in such a way that it would have the same ratio with the extremes. If we arrange 2, 4, and 6 in a line, 4 will have a different ratio with each of the extremes, because the ratio of 4 to 2 is double whereas that of 6 to 4 is sesquialter (1 + ½). Now in order for 4 to be a proportional mean, it would be necessary for the ratio of the first term to the mean to be equal to the ratio of the mean to the third term. Similarly 9, a square number, is contained between the successive unequilaterals 6 and 12, but it does not have the same ratio with the extremes, for the ratio of 9 to 6 is sesquialter (1 + ½), while that of 12 to 9 is sesquitertian (1 + ⅓). It is the same situation with the following unequilaterals.[43]

OBLONG NUMBERS

XVII. An oblong number is a number formed by any unequal numbers of which one is greater than the other, either by one unit, or by two units or by a larger number. Such is 24 which has the value of 6 times 4, and other similar numbers. There are three classes of oblong numbers. Indeed all unequilateral numbers are at the same time oblong so far as they have one side larger than the other; but if all unequilateral numbers are by that fact oblong, the opposite is not true, for the number which has one side longer than the other by more than one unit, is oblong; but it is not unequilateral, since the latter is defined as: a number of which one side is longer than the other by one unit, such as 6, since 2 x 3 = 6.

A number is also oblong when, according to various multiplications, it has one side sometimes longer by one unit, and sometimes longer by more than one unit. Such is 12 which results from 3 x 4

[43] See note III.

and from 2 x 6, so that when the sides are 3 and 4 the number 12 is unequilateral and when the sides are 2 and 6 it is oblong. Finally, a number is again oblong, if, resulting from any kind of multiplication, it has one side longer than the other by more than one unit. 40 is of this sort being the product of 10 by 4, of 8 by 5 and of 20 by 2. Numbers of this kind can only be oblong. The unequilateral number is the one that receives the first alteration after the number formed by equal factors, the first alteration being the addition of one unit given to one of the two sides. This is why the numbers resulting from this first alteration of the sides have been called, with good reason, unequilateral; but those which have a side greater than the others by a quantity larger than one unit have been called oblong numbers because of the greater difference of length between the sides.

XVIII. The planar numbers are the numbers produced by the multiplication of the two numbers representing the length and the width. Among these numbers there are those which are triangular, others which are quadrangular, pentagonal and in general, polygonal.

TRIANGULAR NUMBERS, THE METHOD OF OBTAINING THEM AND OTHER POLYGONAL NUMBERS IN GENERAL

XIX. The triangular numbers are obtained in the manner that we are going to indicate. And first of all the successive even numbers, added to each other, produce the unequilaterals. Thus the first even number 2 is at the same time unequilateral, for it has the value of 1 x 2. If now 4 is added to 2, the sum will be 6, which is again unequilateral, since it has the value of 2 x 3 and the same occurs with following numbers to infinity. But in order to make clearer what we have just said, we will show it in the following way.

Let us suppose that the first even number 2 be represented by two units, 1 and 1. The figure which they form is unequilateral because it is 2 in length and 1 in width. After the number 2 comes the even number 4; if we add the four units to the first two, by placing them around them (at a right angle), we will have the figure of the unequilateral number 6, since its length is 3 and its width 2. After the number 4 comes the number 6. If we add 6 units to the six first ones, by placing them around them (at a right angle), the sum will be 12

and the figure will be unequilateral as its length is 4 and its width 3, and this will continue to infinity through the addition of the even numbers.

```
1 1 1        1 1 1 1
1 1 1        1 1 1 1
             1 1 1 1
```

In turn, the odd numbers added together give the square numbers. Now the successive odd numbers are 1, 3, 5, 7, 9, 11. In adding them in a continuous manner, the square numbers are obtained. Thus unity is the first square number, since 1 x 1 = 1. Next comes the odd number 3. If this gnomon[44] is added to the unity,[45] an equally equal square is obtained, because it has 2 for both length and width. The odd number which comes next is 5. If this gnomon is added to the square 4, a new square 9 is obtained, which has 3 for both length and width. Next comes the odd number 7, which, added to the square 9 gives the square 16, whose length and width are both 4 and so forth to infinity.

```
                     1 1 1 1
1 1      1 1 1       1 1 1 1
1 1      1 1 1       1 1 1 1
         1 1 1       1 1 1 1
```

Likewise, by adding no longer just the evens alone or the odds alone, but the evens and the odds, we will obtain the triangular numbers. The series of the even and odd numbers is 1, 2, 3, 4, 5, 6, 7, 8, 9, 10; in adding them together we form the triangular numbers. The first is unity or monad, because if it is not so in act it is so in power, being the principle of all numbers. If the number 2 is added to it, the result is the triangular number 3. If 3 is added to this triangular number, 6 is obtained, and by adding 4 to this, the result is 10. If 5 is added to this you will have 15, to this add 6 and you will have 21, to which if you add 7 you will have 28 which, augmented by 8, becomes 36. And 36 augmented by 9 becomes 45. Add 10 and you will have 55. And this continues to infinity. Now it is evident that these numbers are triangular according to the figure obtained by adding the successive gnomons[46] to the first numbers.

[44] It is very probable that *gnomon* here has the meaning of *constant* as the succession of odd numbers can be taken as a constant. (Toulis)

[45] The *gnomons* are here the successive odd numbers. See the general definition of gnomon, 1, XXIII.

[46] In this case the gnomons are the natural series of numbers.

The triangular numbers obtained by addition will then be

3, 6, 10, 15, 21, 28, 36, 45, 55

and so forth.

```
1    3      6        10          15            21
1    1      1         1           1             1
    1 1    1 1       1 1         1 1           1 1
          1 1 1     1 1 1       1 1 1         1 1 1
                   1 1 1 1     1 1 1 1       1 1 1 1
                              1 1 1 1 1     1 1 1 1 1
                                           1 1 1 1 1 1

            28                        36

             1                         1
            1 1                       1 1
           1 1 1                     1 1 1
          1 1 1 1                   1 1 1 1
         1 1 1 1 1                 1 1 1 1 1
        1 1 1 1 1 1               1 1 1 1 1 1
       1 1 1 1 1 1 1             1 1 1 1 1 1 1
                                1 1 1 1 1 1 1 1
```

XX. As we have said, the square numbers are produced by the addition of successive odd numbers, beginning with unity or the monad. It so happens that they are alternately odd and even, just as the simple numbers are alternatively odd and even such as:

1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

If now the even and odd numbers are arranged in order, beginning with the unit, it will be seen that, the gnomons, each of which is greater by 2 than the preceding, when added to the square preceding it, will form the next square, as we have shown above. The odd numbers, beginning with the unit, indeed increase by 2. Likewise, the numbers which increase by the addition of 3, always beginning with the unit, form the pentagons. Those which increase by 4 give the hexagons; in such a way that the rate of the gnomons, which gives a polygon, is always less by 2 units than the number of angles in the figure.

There is another order of polygonal numbers, given by the multiple numbers starting from the unit. Indeed, among the multiple numbers starting from unity come the doubles, the triples and so forth, the terms are squares of two by two and cubics of three by three. Furthermore, the following ones of 6 by 6 are both square and cubic; as cubes, their sides are of square numbers, and as squares,

their sides are cubic numbers. Here is how we show that the multiple numbers, beginning with the unit, are squares in two by two and cubes in three by three, and both square and cubic in six by six. Let us arrange several numbers in double progression:

1, 2, 4, 8, 16, 32, 64, 128, 256.

The first double is 2. Next comes 4 which is square, then 8 which is cubic, then 16 which is again square. This is followed by 32, after which comes 64 which is both square and cubic. Next we have 128, followed by 256 which is square; and one could continue this way to infinity.

In the triple progression a similar development of alternate squares is found. Likewise in the quintuple progression and in the other multiple progressions. If two terms are alternatively omitted, it will be found that the remaining terms are both square and cubic.[47] The squares have this property of being exactly divisible by 3, or will become so when diminished by one unit. They are also exactly divisible by 4 or will become so after the subtraction of one unit. The even square which becomes divisible by 3 after having been diminished by one unit, is divisible by 4, which is the case with 4. The square, which becomes divisible by 4 after having been diminished by one unit, is divisible by 3, which is the case with 9.[48] A square can be divisible both by 3 and 4, like 36. Finally, the square which is neither divisible by 3 nor by 4, like 25, admits these two divisors after the subtraction of one unit.[49]

XXI. Among the numbers, the ones which are equally-equal are squares, the others, which are unequally-unequal, are unequilateral or oblong. And in all cases, the products of two factors are planes and those of three factors are solids. They are given the names planar numbers, triangular or square, or solid numbers, and other similar names, not in a proper sense, but by comparison with the space which they seem to measure. Thus 4 is called a square number

[47] Notation in exponents makes these facts evident. Given the progression 1, 2, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}, 2^{11}, 2^{12}...the terms 2^2, 2^4, 2^6....taken two by two, are squares, since the exponents are even; the terms 2^3, 2^6, 2^9...taken three by three are cubic, since the exponent is a multiple of 3; and the terms 2^6, 2^{12}....taken 6 by 6 are both squares and cubic. As squares, their roots 2^3 and 2^6....are cubes and as cubes, their roots 2^2 and 2^4...are squares.

[48] Or, it is the square diminished by one unit which is also divisible by 3, such as the squares of 25 and 49.

[49] See note IV.

because it measures a square space; and for a reason founded on a similar analogy, 6 is called unequilateral.

XXII. Among the planar numbers, the squares are all similar to each other. Among the planar numbers which have unequal sides, those are similar whose sides, or the numbers which comprise them, have the same relationship to one another. Let us take the unequilateral number 6, whose sides, the length and width, are 3 and 2, and another planar number 24, whose sides, the length and width, are 6 and 4. The length of one is to the length of the other as the width of one is to the width of the other, for we have $6 : 3 = 4 : 2$. Therefore the planar numbers 6 and 24 are similar. The same numbers representing lengths at one time can be taken as sides at another for the formation of other numbers; when they are product of the multiplication of two numbers they are called planar and they are solid when they are the products of the multiplication of three numbers.

All cubes are similar, as are the other solids (rectangular parallelopipeds) having proportional sides, in such a way that there is the same relationship between the length of one and the length of the other, the width of one and the width of the other, and finally the height of one and the height of the other.

XXIII. Of all planar and polygonal numbers, the first is the triangular number, just as among planar rectilinear figures the first is the triangle. In the preceding [50] we have discussed the creation of triangular numbers, and we have seen that it consists in adding to the number 1 the natural series of even and odd numbers. Now, all the successive numbers which serve to form the triangular, quadrangular or polygonal numbers, are called "*gnomons*";[51] and the sides of any triangle always have as many units as are contained in the last gnomon added to it. First of all let us take unity, which is not a triangle in fact, as we have already said, but in power, for in being as the seed of all numbers, unity also possesses the faculty of engendering the triangle.

When it is added to the number 2, it gives birth to the triangle whose three sides contain as many units as has the added gnomon (constant) 2, and the entire triangle contains as is many units as contained in the gnomons added together. Because the sum of the

[50] See I, XIX.

[51] Note — See previous explanation of *gnomon* which in all probability is the word "constant" here. *Gnomon* is a guide if we take it literally. (Toulis)

gnomon 1 and the gnomon 2 is equal to 3, so that the triangle is composed of three units, there are two units to each of its sides, that is, as many units as there are of gnomons added together.

To the triangle 3 is then added the gnomon 3, which is greater than the number 2 by one unit, and the whole triangle becomes 6. Its sides each have as many units as there are added gnomons, and the triangle has the value of as many units as the added gnomons contain, for in adding 2 and 3 to unity we arrive at the number 6.

The number 6 augmented by the gnomon 4 gives the triangle of 10 units, the sides of which have 4 units each. Indeed, the gnomon which has just been added is 4 and the whole triangle is composed of units of 4 gnomons, that is 1 + 2 + 3 + 4. The number 10 being augmented by the gnomon 5, gives the triangle 15, each side of which has 5 units, being composed of 5 gnomons, and it is in this manner that the gnomons form the corresponding triangular numbers.

XXIV. Some numbers are called circular, spherical or recurrent. They are those which when squarely or cubically multiplied, that is to say according to two or three dimensions, come back to the number which had been their point of departure. Such also is the circle which comes back to the point at which it began, because it consists of a single line and it begins and ends at the same point. Among solids, the sphere has the same property, because it is described by the revolution of a circle around a diameter, the circle coming back to the position from which it departed. Likewise, the numbers which through multiplication, end with themselves, are called circular or spherical. These numbers are 5 and 6. In fact, 5 x 5 = 25: 25 x 5 = 125; 6 x 6 = 36; and 36 x 6 = 216.

XXV. As we have said[52] the square numbers are created by the addition of odd numbers, that is to say, those which, in starting from unity, increase by 2. That is to say, 1 + 3 = 4; 4 + 5 = 9; 9 + 7 = 16; 16 + 9 = 25.

1	4	9	16	25
1	1 1	1 1 1	1 1 1 1	1 1 1 1 1
	1 1	1 1 1	1 1 1 1	1 1 1 1 1
		1 1 1	1 1 1 1	1 1 1 1 1
			1 1 1 1	1 1 1 1 1
				1 1 1 1 1

[52] See I, XIX.

XXVI. The pentagonal numbers are those which are formed by the addition of the numbers which increase by 3, starting from unity. Their gnomons are thus

1, 4, 7, 10, 13, 16, 19

and the polygons themselves are

1, 5, 12, 22, 35, 51, 70

and so forth. Here is the figure for the pentagonal numbers:

```
1      5        12          22             35

1      1         1           1              1
      1 1       1 1         1 1            1 1
      1 1      1 1 1       1 1 1          1 1 1
               1 1 1      1 1 1 1        1 1 1 1
               1 1 1      1 1 1 1       1 1 1 1 1
                          1 1 1 1       1 1 1 1 1
                          1 1 1 1       1 1 1 1 1
                                        1 1 1 1 1
                                        1 1 1 1 1
                                                    etc
```

XXVII. The hexagonal numbers are those which are formed by the addition of numbers which increase by 4, starting from unity. The gnomons are

1, 5, 9, 13, 17, 21, 25

from which result the hexagons

1, 6, 15, 28, 45, 66, 91

Here are their figures:

```
1      6        15          28             45

1      1         1           1              1
      1 1       1 1         1 1            1 1
      1 1      1 1 1       1 1 1          1 1 1
       1       1 1 1      1 1 1 1        1 1 1 1
               1 1 1      1 1 1 1       1 1 1 1 1
                1 1       1 1 1 1       1 1 1 1 1
                 1        1 1 1 1       1 1 1 1 1
                           1 1 1        1 1 1 1 1
                            1 1         1 1 1 1 1
                             1           1 1 1 1
                                          1 1 1
                                           1 1
                                            1
                                                    etc
```

The other polygonal numbers are composed in the same manner. The heptagons are those which are formed by the addition of num-

bers increasing by 5, starting from unity. The gnomons are:

1, 6, 11, 16, 21, 26

from which result the heptagons

1, 7, 18, 34, 55, 81

The octagons are similarly composed of numbers which increase by 6 starting from unity, the enneagons by numbers increasing by 7 starting from unity , the decagons by numbers increasing by 8. Thus generally in all the polygons, in removing two units from the number of angles, one will have the quantity by which the numbers serving to form the polygon must increase.[53]

XXVIII. The sum of two successive triangles gives a square. This 1 and 3 make 4; 3 and 6 make 9; 6 and 10 make 16; 10 and 15 make 25; 15 and 21 make 36; 21 and 28 make 49; 28 and 36 make 64; 36 and 45 make 81. The triangular numbers which follow, combined together also form squares, likewise the union of two linear triangles gives the figure of a quadrangle.[54]

XXIX. Among the solid numbers, some have equal sides (as when three equal numbers are multiplied with each other); others have unequal sides. Among the latter, some have all sides unequal while others have two sides equal and another unequal. Among those which have two sides equal, some have a larger third side, others a smaller third side.

Those which have equal sides (being equally equal equally), are called cubes. Those, on the other hand, which have all sides unequal, and which are unequally unequal unequally, are called *altars.* Those which have two sides equal and the third smaller than the two others, being equally equal deficients, have been called *plinths* or squares. Finally those which have two equal sides and the third larger than the two others, being equally equal exceedants, are called *beams.*

[53] See note V.

[54] A square number n^2 can be broken down into two triangular numbers, the n^n and the $(n - 1)^n$, so that

$$\frac{n(n+1)}{2} + \frac{(n-1)n}{2} = n^2$$

Thus the square number 25 is broken down into two triangular numbers, the 5^n equal to 1 + 2 + 3 + 4 + 5 and the 4^n equal to 1 + 2 + 3 + 4 and indicated by the figure:

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{array}$$

PYRAMIDAL NUMBERS

XXX. The pyramidal numbers are those which measure pyramids and truncated pyramids. Now a truncated pyramid is (what remains of) a pyramid whose upper part has been taken away. Some have given the name trapezoid (solid) to such a figure, through analogy with the planar trapezoids, since this is what one calls (that which remains of) a triangle of which a straight line parallel to the base has separated off the upper part.[55]

THE LATERAL AND DIAGONAL NUMBERS

XXXI. Just as the numbers have in power relationships with triangles, tetragons, pentagons and other figures, so also we find that the relationships of lateral numbers and diagonal numbers are manifested in numbers according to generative ratios, for these are the numbers which harmonize figures. Therefore since unity is the principle of all figures, according to the supreme generative ratio, it is thus that the relationship of the diagonal and the side is found within unity.

Let us suppose for example two units, one of which is the diagonal and the other the side, because it is necessary that unity, which is the principle of all, be in power the side and the diagonal; let us add the diagonal to the side, and to the diagonal let us add two sides, for that which the side can do twice, the diagonal can do once.[56] From this point on the diagonal becomes larger and the side smaller. Now, for the first side and the first diagonal, the square of the diagonal unit will be less by one unit than the double square of the side unit, for the units are in equality, but one is less by one unit than the double of unity. Let us now add the diagonal to the side, that is to say one unit to unit, and the side will then have the value of 2 units; but if we add two sides to the diagonal, that is to say, 2 units to unity, the diagonal will have the value of 3 units; the square constructed on side 2 is 4, and the square of the diagonal is 9, which is greater by one unit than the double square of 2.

In the same way let us add to side 2 the diagonal 3, the side will become 5. If to the diagonal 3 we add two sides, that is 2 times 2, we will have 7 units. The square constructed on side 5 is 25, and

[55] See note VI.
[56] That is to say that two times the square of the side equals one times the square of the diagonal.

that which is constructed on the diagonal 7 is 49, which is less by one unit than the double (50) of the square 25. Again, if the diagonal 7 is added to the side 5, 12 units are obtained; and if two times the side 5 are added to the diagonal 7, we will have 17, whose square (289) is larger by one unit than the double (288) of the square of 12, and so forth in continuing the addition. The proportion alternates: the square constructed on the diagonal will be sometimes smaller, sometimes larger by one unit than the double of the square constructed on the side, in such a way that these diagonals and these sides will always be expressible.

Inversely, the diagonals compared to the sides in power, are sometimes larger by one unit than the doubles and sometimes smaller by one unit. All the diagonals are then, with respect to the squares of the sides, double, alternately by excess and by lack, the same unit, combined with each of them reestablishing equality, in such a way that the double does not err by excess or by default; indeed, what is lacking in the preceding diagonal is found in excess in the diagonal which follows in power.[57]

PERFECT NUMBERS, ABUNDANT NUMBERS AND DEFICIENT NUMBERS.

XXXII. Furthermore, among numbers, some are called perfect, others abundant, and others deficient. Those are called *perfect* which are equal to (the sum of) their aliquot parts, such as 6. The parts of 6 are in fact its half, 3, its third, 2, and its sixth, 1, which when added together make 6.

This is how the perfect numbers are created: If we arrange the numbers in a progression by doubles, starting from unity, and we add them until we obtain a prime, incomposite number, and if we multiply this sum by the last term added, the product will be a perfect number.[58] Let us then arrange the numbers in a progression by doubles, 1, 2, 4, 8, 16. Let us add 1 and 2, the sum is 3; if we multiply this by the last number added, which is 2, we will have 6 which is the first perfect number (since $1 + 2 + 3 = 6$). If we now add the three successive doubles, 1, 2, and 4, the sum, 7, multiplied by the last number added, 4, gives 28, which is the second perfect number. In fact, as aliquot parts, it has its half, which is 14, its quarter which is 7, its seventh which is 4, its 14th which is 2 and its

[57] See note VII.
[58] Cf. Euclid, *Elements,* IX, 36.

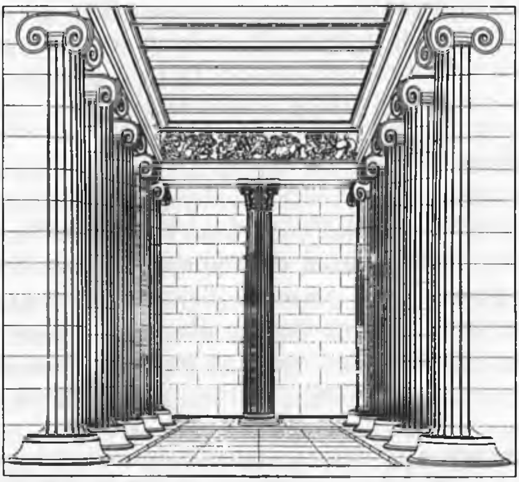
28th which is 1 (and we have $1 + 2 + 4 + 7 + 14 = 28$).

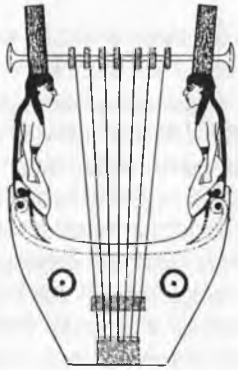
The abundant number is the number whose aliquot parts added together make a sum larger than the original number. Such is 12, half of which is 6, a third 4, a quarter 3, a sixth 2 and a twelfth 1. However, all of these parts added together give the sum of 16, which is greater than the original number 12.

The deficient number is the number whose aliquot parts added together give a sum less than the original number. Such is 8, the half of which is 4, the quarter 2, and the eighth 1. The same is found with the number 10 which the Pythagoreans, nevertheless, call perfect for another reason, which we will speak of in its place.[59]

It is also said that the number 3 is perfect, because it is the first one which has a beginning, a middle and an end; and it is both a line and a surface. It is in fact an equilateral triangular number, each of the sides of which have the value of two units. Finally the number 3 is the first link and the power of the solid, because the idea of the solid rests on three dimensions.

[59] See note VIII.





PART TWO

BOOK CONTAINING THE NUMERICAL THEOREMS OF MUSIC

INTRODUCTION

I. Since it is said that there are consonant numbers, the reason for this consonance cannot be found without arithmetic. This consonance[1] has the greatest power, being truth in reason, felicity in life, and harmony in nature; and this harmony, which is diffused throughout the world, will not be found unless it is revealed first through numbers. This harmony, which is also intelligible, is more easily understood than perceptible harmony. We will therefore consider these two harmonies, learning about the one sensed through instruments, and the intelligible harmony which consists in numbers.

And after having finished our treatise on all mathematics, we will add to it a dissertation on the harmony of the world, and we will not hesitate to relate what our predecessors have discovered, nor to make more widely known the Pythagorean traditions which we
15 have inherited without ourselves claiming to have discovered the least part of it. Desiring then to illustrate to those who wish to study Plato further what has been transmitted to us by our predecessors, we have judged it necessary to compose this compendium.

[1] The word symphony should not be taken with its modern meaning of the present, i. e. a musical work played by a large symphony orchestra, etc. In those days symphony meant only one thing — agreement, accordance, or literally symphony from the word Συμφωνία. Later the word consonance had to be employed by English/French peoples because they thought that the word symphony would be best suited to represent the idea of a work of music, which is a misuse.

WHAT IS SOUND AND ENHARMONIC SOUND

II. Thrasyllus, in discussing the perceptible harmony of instruments, defined sound as a tension of the enharmonic sound. Now sound is said to be enharmonic when, if it is high-pitched, it can have a still higher pitch, and if it is low pitched, it can become still lower so that the same sound is found to be median as well. If then we imagine a sound which is higher than all others, it would not be called enharmonic, and for that reason, one never regards the violent noise of the thunderbolt, the injuries from which are always fatal, as enharmonic. As Euripides the poet says,

> And the blow of the thunderbolt has made many a victim
> Without wounds that bleed.

Similarly, if the sound is so low pitched that there can be none lower, it will no longer be a sound because it will no longer be enharmonic. It is therefore neither every voice nor every tension of sound which would be called sound, but only an enharmonic sound, like that which comes from the *mese,* the *nete* or the *hypate.*[2]

WHAT IS INTERVAL AND HARMONY

III. The interval is defined as the relationship of sounds among each other, such as the fourth, the fifth and the octave. And a certain grouping, such as the tetracord, the pentacord and the octacord, is called a system of intervals.

IV. Harmony is the order of systems; such are the Lydian, the Phrygian, the Dorian. As for the sounds, some are high, others low and others medium pitched. The high sounds are those which give the *netes,* the low sounds those which give the *hypates* and the medium sounds those which give the intermediate strings.

V. Among the intervals, some are consonant, others dissonant. The consonant intervals are antiphonic, such as the octave and the double octave, or paraphonic such as the fifth and the fourth. The dissonants, on the contrary, are intervals of sounds juxtaposed to each other such as the tone and the diesis (or half-tone). The antiphonic intervals or opposed sounds are consonant, because depth

[2] In the octacord or eight-stringed lyre, the *nete* gave the highest sound, the *hypate* the lowest. These two sounds correspond to the two *mi* of the same octave, the mese corresponds to the *la*.

opposed to height produces a consonance, and paraphonic intervals are consonant because the sounds are neither in unison nor dissonant, but there is a similar, perceptible interval. The sounds are dissonant and not consonant whose interval is a tone or a diesis; because the tone and the diesis are the principle of consonance but they are not consonance itself.

ON CONSONANCES [3]

VI. Adrastus, the student of Aristotle, in his well known treatise *On Harmony and Consonance,* says: Likewise, in discourse, whether written or spoken, verbs and nouns are the most important parts; the essential parts of verbs and nouns are the syllables composed of letters; and letters are the primary signs of language, being elementary, indivisible and the shortest element, since discourse is composed of letters and, in the end, resolves into letters. In the same way, that which makes up the principle part of sound and of all melody are the systems called tetrachords, pentachords and octachords, which are composed of intervals which are themselves composed of sounds, these sounds being the primary and indivisible elements of which all melody is composed, and into which it definitively resolves itself.... The sounds differ from each other through tensions, some being higher, others lower. These tensions are defined in different manners.[4]

Let us examine, in this regard, the opinion attributed to the Pythagoreans. Every modulation and every sound being a voice, and every voice being a noise and noise being a percussion of air which is broken by it, it is evident that in immobile air neither noise nor tone would exist. On the contrary, when the air is struck and put into movement, sound is produced; it is high if the movement is rapid, low if the movement is slow; strong if the movement is violent, weak if the movement is faint. The speeds of these movements occur according to certain relationships, or according to none.

The speeds without relationship result in sounds without relationship and dissonance, and which, properly speaking, do not deserve the name of sounds, and would more rightly be called noise. On the other hand, one must regard as true sounds, belong-

[3] See footnote 1.

[4] The tension of a sound is now called the height.

ing to modulation, those which have a particular relationship to each other, whether multiple or superpartial[5] or simply of number to number. Of these sounds, some are concordant only, others are symphonic according to primary ratios and best known multiples, and according to superpartial ratios.

They form a consonance with each other when a sound which is produced by a string of an instrument, causes the other strings to resonate by means of a certain affinity, a kind of sympathy; and also, when two sounds being produced at the same time result in a mixed sound which has a sweetness and a quite particular charm. Among these concordant sounds, the first to form such a concordance are the fourths that form a consonance with themselves, and for this reason are called fourths. The fifths in turn give the fifth.

Next come the eighths which include these two consonances and which we call the diapason (octave). Indeed, on the eight-stringed lyre, the first sound is found to be the lowest, and is called the hypate. It is in accord by opposition with the last and highest which is that of the nete, with which it has the same consonance. And when, music having made progress, instruments were given a greater number of strings, rendering more multiple sounds — a large number of sounds, more of them high than low, having been added to the ancient eight — the denominations of the ancient consonances, the fourth, the fifth and the octave, were nevertheless retained.

However, several other consonances have been found: added to the consonance of the octave are smaller, larger or equal intervals, and the sum of the two results in a new consonance, such as the octave and the fourth, the octave and the fifth, and the double octave. And if any of the preceding consonances is added again to the octave, the double octave ana the fourth, for example, is produced, and so forth, as far as a sound perceptible to the ear can be produced. There is, in fact, a certain range that the sound can travel, beginning with the lowest sound and going up to the highest and inversely, a range which is greater in some, less in others.

This series of modulations is not situated by chance, nor without art and according to a single mode, but following certain determined modes which must be observed in different types of melody. For in the same way that in discourse, whether spoken or written, not just any letter combined with any other letter produces a syllable or a word, likewise in melody; it is not the combination of just

[5] The superpartial or sesquipartial relationship is that in which the antecedent is greater than the consequent by one unit, like that of 3 to 2, 4 to 3 and generally that of n + 1 to n.

any sounds that produces the well ordered sound, or which, in its turn, produces the interval appropriate to modulation; but it is necessary that this combination take place, as we have just said, following the law of defined modes.

THE TONE AND THE HALF TONE

VII. The easiest part to appreciate and the measure of what is called the range of sound and of any sound interval, is called tone, in the same way that the principle measure of space which surrounds all moving bodies is called the cubit. The interval of the tone is very easy to distinguish through the difference between the first and best known consonances: the fifth is greater than the fourth by one tone.

VIII. The half-tone is not designated as such because it is the half of the tone in the way that the half-cubit is the half division of the cubit as maintained by Aristoxenes; but because it is a musical interval less than the tone, in the same manner that we call certain letters demi-vowels, not because half of a sound is indicated, but because it does not completely compose the sound itself. It can actually be demonstrated that the tone, considered in the sesquioctave ratio ($9/8$), cannot be divided into two equal parts, any more than can any other sesquipartial, since 9 is not divisible by 2.

ON THE DIATONIC, THE CHROMATIC AND THE ENHARMONIC FORMS OF MELODY

IX. When a voice which is modulated within the limits of its range goes from a lower sound to a higher sound, producing the interval of a half-tone, it then, going through the interval of a tone, passes to another sound, and continuing to modulate, can not attain any interval other than that of a tone.† This produces another melodic sound which is apt to modulation, and this higher consonant sound will give, together with the first, the consonance of the fourth.

A modulation of this type is called the tetrachord system, which is composed of three intervals: half-tone, a tone, and another tone, and these four sounds, the extremities, that is to say the lowest and the highest, form a consonance. This consonance, which we have

† Theon is expounding the tetrachord theory of Aristoxenus.

said is called the fourth, is then composed of two tones and a half-tone. This type of modulation is called diatonic, either because, in the ordinary sense, it goes up by two tones, or because of the vigour and firm character which it manifests.

X. When the voice produces a first sound, and passing through a half-tone, rises to a higher sound, and then passes on to a third, again passing through a half-tone, and proceeding with modulation, produces another after this, it can observe no other interval than that of a complete trihemitone, the complement of the first tetrachord, and it can produce no other sound than that which limits this tetrachord in rising towards the higher sounds, and which, with the lowest, gives the consonance of the fourth. This modulation is made then by a half-tone, followed by a half-tone and a complete trihemitone. and this type of modulation is called chromatic, because it diverges from the first, and it changes color, expressing sad affections and violent passions.

XI. There is a third type of modulation which is called enharmonic. This is the type in which, starting from the lowest note, the voice modulates the tetrachord, by progression through a diesis, then another diesis and a double tone.

CONCERNING THE DIESIS

XII. The disciples of Aristoxenes call the quarter-tone or half of the half-tone the diesis minor, and consider it the smallest appreciable interval. The Pythagoreans call what is now called the half-tone, the diesis. [6] Aristoxenes says that the enharmonic type is so-called because it is the best, and this caused him to give it the name which applies to everything well-ordained. This modulation is very difficult, and as he said himself, it requires much art and study and is only acquired through long practice. The diatonic type, on the other hand, is simple, noble and more natural, as taken by Plato. [7]

[6] Now, that is to say, at the beginning of the second century.

[7] Plato, according to Macrobius, also assigned the diatonic type to the harmony of the spheres:...*"diatonum (genus) mundanae| musicae doctrina Platonis adscribitur."* Macrobius, *In somnium Scipionis*, II, 4.

TYPES	INTERVALS		
Diatonic	half-tone	tone	tone
Chromatic	half-tone	half-tone	trihemitone
Enharmonic	diesis diesis		ditone

THE DISCOVERY OF THE NUMERICAL LAWS OF CONSONANCES

XIIa. It is Pythagoras who appears to have first found that the consonant sounds are in relationship to one another.[8] The sounds which produce the fourth are in sesquitertian ($4/3$) relationship to each other; those which produce the fifth are in sesquialter ($3/2$) ratio; those which produce the octave and fourth are in the relationship of 8 to 3 which is polyepimer, since it is equal to $2 + 2/3$. The sounds which give the octave and fifth are in triple ratio, and those which give the double octave are in quadruple ratio. Among the other concordant sounds, those which give the tone are in sesquioctave ($9/8$) ratio, and those which give the half-tone, but which then were called diesis, are in the relationship of the number 256 to the number 243.[9] It is Pythagoras, we say, who appears to have discovered these relationships, through the length and thickness of strings as well as through the tension applied to them by the turning of a peg, or by the more familiar method of suspending weights from them, and for wind instruments by the diameter of the cavity, by greater or less intensity of breath, or by the weight of disks or the level of the water in vases. Whatever the method chosen from among those we have just cited, the consonance will follow the indicated relationship, all things otherwise being equal.

For the moment, let us content ourselves with the demonstration which, in what is called the harmonic canon, is obtained by string-lengths. If we divide a stretched string into four equal parts on the harmonic canon, the sound produced by the whole string will form, with that produced by three parts of the string, the accord of the fourth, and the relationship is sesquitertian. With the sound pro-

[8] Cf. Chalcidius, *In Timaeum Platonis*, XLIV, p. 191, Paris, Didot.

[9] The relationship of 256 to 243 is also called leimma and is the excess of the fourth over the double tone, that is $4/3 : (9/8)^2 = 4/3 \times 64/81 = 256/243$.

duced by two parts or half of the string, it will form the accord of the *octave,* and the relationship is double. With the sound produced by a quarter of the string, the accord of the double octave will be produced, since the relationship is quadruple.

Furthermore, the sound produced by three parts of the string will give, with the sound produced by half of the string, the consonance of the fifth, for the relationship is sesquialter and, with regard to the sound produced by a quarter of the string, it will give the consonance of the octave and a fifth, for the relationship is 3. If we divide the string into 9 equal parts, the sound produced by the whole string will give, with the sound produced by 8 parts of the string, the interval of one tone, for the relationship is sesquioctave.

The quaternary, 1, 2, 3, 4, includes all the consonances, since it contains those of the fourth, the fifth, the octave, the octave and fifth, and the double octave, which are the sesquitertian, sesquialter, double, triple and quadruple ratios (that is to say, $4/3$, $3/2$, 2,3, and 4).[10]

Some prefer to obtain these consonances by weights, others by lengths, and others by numbered movements, and yet others by the capacity of the vessels. It is said that Lasus of Hermione and the disciples of Hippasus of Metapontus, the latter of the Pythagorean school, observed the rapidity and slowness of the motions of liquids created in vessels, which is helpful in the calculation of the consonances into numbers. Taking several similar vessels of the same capacity, one was left empty and the other filled half way with a liquid, then they were both struck, thus obtaining the consonance of the octave.

Again leaving one vessel empty and filling the other up to one quarter, then striking them, the consonance of the fourth was obtained. For the accord of the fifth, a third of a vase was filled; the relationship of the empty spaces was 2 to 1 for the octave, 3 to 2 for the fifth, and 4 to 3 for the fourth.

The same relationships as we have seen are obtained by the division of strings. One does not use a single string in every case as in the harmonic canon, but two equally stretched strings in unison. Half of one of these strings was plucked by pressing in the middle of it with a finger, obtaining with half of one and the entire other

[10] This is the tetractys (in Greek Τετρακτύς) and it was the name Pythagoreans gave to the number 10 since it is the product of 1 + 2 + 3 + 4 = 10. (Toulis.)

string, the octave. When they were plucked in thirds only, the two other thirds and the entire string gave the accord of the fifth. In the same way, to obtain the consonance of the fourth, a quarter of one of the two cords was plucked, leaving the other one whole.

A similar experiment has been made on the flute and the same relationships were found. Those who have measured consonances with weights, suspended the weights from two strings in the relationships which we have mentioned and that had been obtained by the length of the strings, in determining the consonances of these cords.

XIII. Tone is the resting of the sound on a single intonation; for it is said that the sound should always be similar to itself and not admit the least difference, nor be composed of different tensions of lowness or highness. But sounds are in part high pitched and in part low pitched, this is why among tones, one, the high, is rapid, and the other, the low, is slow. If then one blows into two pipes, of equal thickness and of equal diameter, pierced in the manner of a flute, one of which is two times longer than the other, the air which escapes from the tube which is two times less long has a double speed and results in the consonance of the octave, the lower tone coming from the longer pipe and the higher tone coming from the shorter pipe.

The cause of this must be attributed to the velocity and to the slowness of the motion, and this cause produces the same consonances in a single flute, due to the distance of the holes. In fact, if with a flute divided into two equal parts, one blows into the whole flute, then up to the hole which divides it into two parts one will hear the consonance of the octave. If the flute is divided into three, two thirds being taken from the tongue end and one third from the far end, and one blows into the whole flute and into the two thirds, one will hear the accord of the fifth. If it is divided into four, and one takes three parts at the top and one at the bottom, in blowing into the whole flute and into the three quarters, one will have the consonance of the fourth.

The schools of Eudoxus and Archytas thought that the relationship of consonances could be expressed by numbers; they also recognized that these relationships express motion, a rapid motion corresponding to a high pitched sound, because it strikes and penetrates the air in a more continuous and quick manner, and a slow motion corresponding to a low pitched sound, because it is more retarded.

This then is what was essential for us to relate concerning the discovery (of the numerical laws) of consonances. Let us return now to what Adrastus has said on the subject of these instruments which were prepared according to certain relationships for the purpose of discovering the consonances. He says, in fact, that we judge the size of the intervals with our ear and that the ratios confirm the testimony of our senses. We will explain presently how the sounds which have a half-tone interval between them, as we have described them, are in the relationship of 256 to 243.

THE RELATIONSHIP OF THE ADDITION AND SUBTRACTION OF CONSONANCES

XIIIa. It is evident that the compositions and the divisions of consonances have the same relationship and harmony to one another as the compositions and the divisions of the numbers which measure the consonances, as we have already explained. Thus the octave is composed of the fifth and the fourth and is divided into the fifth and the fourth. Now the ratio of the octave is double, that of the fourth is sesquitertian ($4/3$) and that of the fifth is sesquialter ($3/2$). It is clear that the ratio 2 is composed of $4/3$ and $3/2$ and is resolved into the same numbers. Thus 8 is $4/3$ of 6 and 12 is $3/2$ of 8, and 12 is the double of 6: one has the numbers 6, 8, 12. Likewise, the ratio '2' of 12 to 6 breaks down into two, the sesquitertian relationship ($4/3$) of 12 to 9, and the sesquialter relationship ($3/2$) of 9 to 6.

As the fifth surpasses the consonance of the fourth by one tone, since it is composed of three tones and a half, the tone being in the sesquioctave relationship ($9/8$), one finds that the sesquialter relationship ($3/2$) also surpasses the sesquitertian relationship ($4/3$) by the sesquioctave ratio ($9/8$). In fact, if from a sesquialter ratio, such as 9 to 6, one subtracts the sesquitertian ratio of 8 to 6, the remainder is the sesquioctave ratio of 9 to 8. [11] Likewise if one adds to this the sesquitertian ratio of 12 to 9, the sesquialter ratio of 12 to 8 will be completed. [12]

As the consonance of the octave is in double ratio and the consonance of the fourth is in sesquitertian ratio ($4/3$), the sum of the two give the ratio of 8 to 3, since 4 is to 3 in the sesquitertian relationship and the double of 4 is 8. [13]

[11] That is $9/6 : 8/6 = 9/8$.

[12] That is $9/8 \times 12/9 = 12/8 = 3/2$.

[13] That is $2 \times 4/3 = 8/3 = 2 + 2/3$.

The octave plus a fifth being in triple ratio, the sesquialter relationship added to two indeed gives this ratio, for the relationship of 9 to 6 is sesquialter and the relationship of 18 to 9 is double, which gives the triple ratio for the relationship of 18 to 6.[14]

$$\underbrace{\underbrace{8 \qquad 4}_{\text{octave} = 2}\underbrace{\qquad 3}_{\text{fourth} = 4/3}}_{\text{octave and fourth} = 2 + 2/3} \qquad \underbrace{\underbrace{18 \qquad 9}_{\text{octave} = 2}\underbrace{\qquad 6}_{\text{fifth} = 3/2}}_{\text{octave and fifth} = 3}$$

The double octave is similarly in quadruple ratio, since it is composed of two double ratios: the double of 6 is 12 and the double of 12 is 24 which is the quadruple of 6; or rather, according to what we have said at the beginning, the triple ratio added to the sesquitertian ratio gives the quadruple ratio. However, the ratio of the octave and the fifth is 3, and that of the fourth is sesquitertian (4/3) and it is of both that the double octave is composed. It is therefore correct that the quadruple consonance is seen here, since the triple of 6 is 18, 4/3 rds of which are 24 which is the quadruple of 6. In the same way, the relationship of 8 to 6 is sesquitertian and the triple of 8 is 24, which is the quadruple of 6. One can push these notions as far as one likes, the same relationships will always be found resulting in the composition of consonances.[15]

Plato carried the diatonic type and the extension of this system up to the fourth octave with an additional fifth and one tone.[16] If it is objected, says Adrastus, that it is not necessary to push this calculation so far, since Aristoxenus limited the extension of the diagram which represents the different modes to the double octave and the fifth, and the moderns have the pentedecacord (the 15 string lyre) whose most considerable extention contains only the double octave (with one additional tone), I respond in answer to this that these latter, considering only the practical point of view, have determined things in this manner because they were persuaded that even those who compete for the singing prize cannot raise their voices above these limits, and that besides, the listeners would no longer be able to distinguish easily the sounds.

Plato, on the contrary, considering the natural things and the soul, which are necessarily composed of harmony, extends the

[14] That is $9/6 \times 18/9 = 18/6 = 3$.

[15] See note IX.

[16] See note X.

calculation up to the solid numbers (8 and 27) and joins the terms by two means, in order to be able to embrace completely everything that the solid body of the world is made up of; and he extended harmony to this point, which by nature, can go on to infinity.

In addition, he says that it is suitable to attribute the largest numbers to the lowest sounds, although this would not appear to be so for certain tensions, for example to the tension which is made by the suspension of weights. Indeed, of two strings equal in length and thickness and similar in all other ways, the one which sustains the greater weight will produce the highest pitched sound because of the greater tension, since the greatest weight, producing a very strong tension, consequently gives a greater force to the higher sound by itself which has, according to this, a force less than the suspended weight. It is evident, on the contrary, that a lower sound possessing by itself a greater force than the suspended weight, is sufficient in itself to retain its own harmony and consonance. Because of this, the largest number must be attributed to the greatest force. This is in agreement with the train of thought, for the lengths and thicknesses of the strings, slowing the movement, cause them to be weak and prevent them from vibrating easily and from rapidly striking the surrounding air.

It is then evident that the lowest sounds have their appropriate force according to the largest number. The same thing is found with the wind instruments, for in these instruments the lowest sounds result from their length and the size of the holes which allow for the release of a greater quantity of air which is subsequently put into motion. They also, by the name of Zeus, result from the diminution of the breath, as in the trumpet and in the organ of the voice in which the feeble and tempered sounds have (also) a greater force of their own.

The first of all the consonances, says Plato, is the fourth, since it is through it that the others are found; the fifth is only separated from the fourth by the interval of one tone.

THE LEIMMA

XIV. The tone can be defined as the interval which separates the fifth from the fourth. One finds that the octave is the sum of the fourth and the fifth, since it is composed of these two consonances.

The ancients took the tone as the first interval of the voice, without taking into account the half-tone or the diesis. They found that the tone is in sesquioctave ratio (9/8). They demonstrated this with disks, vessels, strings, pipes, suspended weights and in several other ways. It was always the relationship of 8 to 9 which allowed the ear to discern the interval of a tone. The first interval (contained in the fourth) is thus the tone; the voice, in reaching this interval, gives the ear a sensation of something fixed and well determined. The ear can again grasp with precision the following interval. As for the interval called the half-tone, which comes after this, some say it is a perfect half-tone, and others that it is a leimma (a remainder). The consonance of the fourth which is in sesquitertian ratio (4/3), is not then completed by a tone, that is to say by a sesquioctave interval (9/8).

Everyone agrees that the interval of the fourth is greater than two tones and less than three tones. Aristoxenus says that it is composed of two and a half perfect tones, while Plato says that this interval is two tones and a remainder (leimma), and he adds that this remainder (leimma) does not have a name, but that it is in a number to number relationship, which is that of 256 to 243 and the difference in this ratio is the number 13.[17]

This is the method which was used to find this relationship: it is known that the first term would not be 6, since 6 is not divisible by 8 and one must take 9/8 ths of it. Neither would it be 8, for if 9/8 ths of 8 are 9, it would not follow to then take 9/8 ths of 9, and it is necessary to take 9/8 ths of 9/8 ths, since the fourth which is in sesquitertian relationship is greater than the double tone. We take therefore the bottom sesquioctave 8 and 9; now 8 multiplied by itself, gives 64 and 9 x 8 gives 72; finally, 9 multiplied by itself, gives 81. We have then (8, 9) 64, 72, 81. If now each of these numbers is multiplied[18] by 3 we have 64 x 3 = 192; 72 x 3 = 216; 81 x 3 = 243; so that we have (8, 9) 64, 72, 81, 192, 216, 243. After 243 let us take 192 x 4/3, or 256, and we will have the series of the following terms:

the bottom sesquioctaves	8,9
the second sesquioctaves	64, 72, 81,
the third sesquioctaves	192, 216, 243.

[17] Cf. *Timaeus* 36B., Plutarch, *On the Creation of the Soul in the Timaeus* 16, 17. *Macrobius*, *Commentary on the dream of Scipio*, II, 1.

[18] One multiplies by 3 in order to be able to take 4/3 rds of the first term to obtain the number which corresponds to the consonance of the fourth.

If $4/3$ rds of 192, or 256, is added, the consonance of the fourth ($4/3$) will be completed by two tones and the leimma of which we have spoken.

$$\underbrace{\underbrace{192 \qquad\qquad 216}_{\text{tone} = 9/8}\ \underbrace{\qquad\qquad 243}_{\text{tone} = 9/8}\ \underbrace{\qquad\qquad 256}_{\text{leimma} = 256/243}}_{\text{fourth} = 4/3}$$

There are some who chose the number 384 for the first term, in order to be able to take the series of $9/8$ ths two times.† They multiply the term 6 by 8, which gives 48, and in multiplying this number again by 8, they have the product 384, $4/3$ rds of which is equal to 512. Between these two terms are found two sesquioctaves; for 384 x $9/8 = 432$ and 432 x $9/8 = 486$ which, with 512, gives the relationship of the leimma.

$$\underbrace{\underbrace{384 \qquad\qquad 432}_{\text{tone} = 9/8}\ \underbrace{\qquad\qquad 486}_{\text{tone} = 9/8}\ \underbrace{\qquad\qquad 512}_{\text{leimma} = 256/243}}_{\text{fourth} = 4/3}$$

Some say that these numbers are not conveniently selected, considering that the excess of the fourth term over the third is not 13, the number that Plato said must be the leimma. But nothing prevents us from finding in other numbers the same relationship which exists between 256 and 243, for Plato did not take a determined number but only that ratio of the number. Now the relationship which exists between 256 and 243 is the same as that between 512 and 486, since 512 is the double of 256 and 486 is the double of 243. It is manifest that this interval of the numbers 256 and 243, of which the difference is 13, is less than the half-tone, because the tone being $1 + 1/8$, the half-tone will be half of $1 + 1/8$, that is to say $1 + 1/16$. Now, $13/243$ is a relationship which is less than $1/18$, a relationship which is itself less than $1/16$.[19]

It is, however, impossible to divide the ratio $1 + 1/8$ into two equal parts, although some believe it possible judging this question not

[19] The half of the tone ($1 + 1/8$) is not $1 + 1/16$ but $\sqrt{9/8}$, hence not a *number* in Theon's sense. See note XI.

† In the complete octave scale there are two consecutive tetrachords. See Pg. 148, strings 4 to 11, and the related numbers in Table II on pg. 150 for the standard Greek octave.

by reasoning but by ear; the base of the sesquioctave interval being the relationship of 9 to 8.[20] The difference of the terms which is the unit , is assuredly not divisible.

XV. If it is asked, concerning the leimma, to which consonance it belongs, we would say that it is necessary to consider it as belonging to the fourth; for it is the leimma which makes the fourth less than two and a half perfect tones.[†]

However, here is how the tone has been found. The fourth being in the relationship of $4/3$, and the fifth in the relationship of $3/2$, the first number divisible by both 2 and 3 has been taken. This number is 6, of which $4/3$ rds equals 8 and $3/2$ nds equals 9. We have 6, 8, and 9 and the excess of the interval $3/2$ over the interval $4/3$ is $9/8$, since 9 is $9/8$ ths of 8. This tension was given the name of tone.

XVI. It is evident that the tone cannot be divided into two equal parts. To begin with, the base sesquioctave $9/8$ has the indivisible unit as the difference between its terms; and then, this interval being expressed by whatever numbers, the difference between the terms cannot always be divided into two equal parts: thus the difference of 27 between the terms of the relationship of 216 to 243 is not susceptible to division into two equal parts, but into two numbers which are 13 and 14, for unity is not divisible. Sometimes we grasp the tone by the operation of the intelligence, and sometimes we look for it in numbers and intervals, and sometimes, finally, we perceive it by the ear in sounds, and we know that it can never be divided into two equal parts, neither in numbers, as we have just shown, nor in perceptible and visible intervals.

It is as in the harmonic canon: the bridge of a stringed instrument which is perceptible has, regardless of its construction, a certain width, and cannot be so thin that in partitioning the tone, it would intercept absolutely nothing from the end of the first part and from the beginning of the second. Thus there will always be a certain part of the tone which will be absorbed. In partitioning, there are then three things: the two divisions and the part subtracted by the bridge. Through the act of the division itself, a part of that which is divided is found to be destroyed, as the product of the cut when one cuts something with a saw. As in certain tangible things some particles are lost, it is the same in all other things, even when our senses do not give us evidence of it. When a cut is made, some small part

[20] The base of a relationship is this relationship reduced to its most simple expression. See II, XXIX, pg. 53.

† The two and a half perfect tones of Theon are not those of Aristoxenus (pg. 44) for Theon is making a fine distinction Aristoxenus considered irrelevant to the ear.

is permanently lost.

If, for example, before dividing the shaft of a lance or a reed or any other long object, you measure and you next divide it into several parts, you will find the length of all the parts put together is less than the length of the object before division: Likewise, if you divide a string into several parts and you cut it, you will find that after the cutting the development will be less, and if you wish to once more stretch all the parts, you will not be able to prevent, in adjoining them by the ends, a part of the string being lost. This is why the two half-tones will never be complete.

And neither in sounds does one find the splitting of the tone into two equal parts:† for if, after having heard a tone followed by another tone, I produce two half-tones instead of one single tone, by three emissions of the sound, going up two intervals, the third sound is higher than the second and it is one tone higher than the first, while it seems to be above the second by only a half-tone; but this half-tone is neither equal nor similar to that which is found between the first sound and the second, the lowest cannot be similar to the highest, and it is in vain that we would wish to reproduce the same sound twice by splitting the sound. We would give some resonance, but there will always be a difference, although imperceptible to the ear.

It is as if one wished to make two punctures perfectly similar, or to pluck a string twice in the exact manner; there will always be more or less a difference of force applied. The same thing will occur if one wishes to plunge his finger the same way twice into a liquid, or into ink or honey or bitumen, in trying to retain the same quantity.

With regard to the ideal tone, one might conceive that it could be divided into two equal parts.

XVII. We should speak now of the harmony which is contained in numbers and to explain what the term is which, in everything, shows the property of what is spoken of, for example, number, size, power, mass, gravity.

ON THE VARIOUS MEANINGS OF THE WORD λογός (logos)

XVIII. The word λογός is taken to have several meanings by the students of Aristotle; the language which the moderns call oral,

† The argument here depends on knowing that the first semitone is 256 : 243 and the second is $\frac{9}{8} \div \frac{256}{243} = \frac{2187}{2048}$

and mental reasoning without the emission of voice are both designated in this way. The relationship of proportion or ratio is also called this, and it is in this sense that it is said that there is a relationship of one thing to another thing. It also has the meaning of the explanation of the elements of the Universe: the recognition of things which honor and are honored, and it is in this sense that we say: taking account of something, or not taking account of it (word of honor). The calculation of bankers is also called λογός, as is the discourse of Demosthenes and Lysias in their written works (speech); the definition of things which explains their essence (reason), since it is to this that it applies; the syllogism and induction; the tales of Lybius, and fables. The name λογός is also given to the eulogy, to the tale and to the proverb. The ratio of form is also called this as well as the seminal ratio and many others.

But, according to Plato, the word λογός is used with four different meanings: for mental thought without words; for discourse proceeding from the mind and expressed by the voice; for the explanation of the elements of the universe; and the ratio of proportion. It is this ratio that we propose now to seek.

THE RATIO OF PROPORTION

XIX. The ratio of proportion of two terms of the same species is a certain relationship that these terms have to each other, such as the double, or the triple. It is impossible, says Adrastus, to find a relationship between two things which are not of the same species: thus one cannot compare the cubit (a measure of length) and the mine (a measure of weight), the chenice (a measure of volume for dry things) and the cotyle (a measure of volume for liquids), the white and the sweet or the hot; but one can compare things of the same species with one another, such as lengths with lengths, surfaces with surfaces, solids with solids, weights with weights, liquids with liquids, fluids with fluids, dry things with dry things, numbers with numbers, time with time, motion with motion, and sound with sound, essence with essence, color with color, indeed all things of the same species.

XX. We call 'terms' homogenous things or things of the same species, taken for the purpose of being compared with each other. When we examine what relationship exists between the talent and the mine, we are saying that they are terms of the same species,

because they are both weights. It is the same with other homogenous things.

XXI. The proportion is the relationship of ratios with each other as 2 is to 1 and 8 is to 4.

XXII. Relationships are greater, lesser, or equal. The equal relationship is *one* and always the same, and it prevails over all others as being elementary. Such are the relationships which compare the same quantity, such as 1 compared to 1, or 2 to 2, 10 to 10, 100 to 100. Among the relationships that are larger than unity, some are multiples (that is to say integers), others are sesquipartials, and others are neutral. Among relationships that are less than unity , some are submultiple, others are subsesquipartial and others are neutral. Among these ratios, some represent consonances, and the others are foreign to it.

The multiple ratios which represent the consonances are the double ratio, the triple ratio and the quadruple ratio: the sesquipartial ratios are the sesquialter ratio ($^3/_2 = 1 + {}^1/_2$), and the sesquitertian ratio ($^4/_3 = 1 + {}^1/_3$). Among the neutral ones, there is the sesquioctave ratio ($^9/_8 = 1 + {}^1/_8$) and the relationship of 256 to 243. Opposite to these ratios are the sub-double ($^1/_2$), the sub-triple ($^1/_3$), the sub-quadruple ($^1/_4$), the subsesquialter ($^2/_3$), the sub-sesquitertian ($^3/_4$), the sub-sesqioctave ($^8/_9$) and the relationship of 243 to 256.

The double ratio, as we have seen above, is found in the consonance of the octave [21], the triple ratio in the consonance of the octave plus the fifth, the quadruple ratio in the double octave, the sesquialter ($1 + {}^1/_2$) in the fifth, the sesquitertian ($1 + {}^1/_3$) in the fourth. As for the sesquioctave ratio ($1 + {}^1/_8$), it is one tone, and the relationship of 256 to 243 is a leimma. It remains the same for the inverse relationships. Among the neutral ratios are the sesquioctave ratio ($1 + {}^1/_8$) and the ratio of 256 to 243 which are not consonances but are also not beyond consonance, since the tone and the leimma are the principles of consonance and have the virtue of completing it without, however, being consonances. [22]

In arithmetic there are ratios of numbers, not only multiples and superpartials, but also the epimer and the polyepimer and other ratios which we will clearly explain later on. The fourth is com-

[21] Cf. II, XII and XIII.

[22] Cf. II, V.

posed of two tones and a leimma, the fifth of three tones and a leimma, the octave of one fifth and one fourth; but the relationships of proportion must precede them.

Thus according to the principles of arithmetic, as Adrastus teaches, there are multiple relationships, while others are sesquipartial, and others epimer, others multisuperpartial, others polyepimer; others are neutral, and among the relationships smaller than unity, there are the sub-multiples, while others are sub-sesquipartial; and the others are inverse relationships larger than unity.

XXIII. The relationship is multiple when the larger term contains the smaller several times, that is, when the smaller term exactly measures the larger without there remaining any part left over. The larger term is given the name of the number of times the smaller measures it: if, for example, it measures it two times, the relationship is double; if it measures it three times, the relationship is triple; if it measures it four times the relationship is quadruple; and so forth. Reciprocally, the smaller term as part of the larger, receives a denomination corresponding to the multiple ratio: it is called the half of the double term, the third of the triple term... and the ratio is called half, third, and so forth.

THE SUPERPARTIAL OR SESQUIPARTIAL RELATIONSHIP

XXIV. The relationship is called sesquipartial when the larger term contains the smaller term plus a part of the smaller term one time, that is to say when the larger term is greater than the smaller by a certain quantity which is a part of it. Thus the number 4 is sesquipartial in relationship to 3, because it surpasses it by one unit which is a third of 3. Likewise 6 surpasses 4 by 2 units which is half of 4.

Each sesquipartial relationship is given, after the designation of the fraction, a particular denomination. Thus the one which is greater than unity by half of the smaller term, such as $^3/_2$ and $^6/_4$, has been called sesquialter since the larger quantity contains the whole of the smaller plus a half of it. 3 indeed contains one times 2, plus unity which is half of 2; 6 contains one times 4, plus 2 which is half of 4. The relationship which is greater than unity by one third of the smaller term, such as $^4/_3$, is called sesquitertian, the one

which is greater than unity by a fourth, such as $5/4$ or $10/8$, is called sesquiquartan, and in continuing in the same manner, the relationships called sesquiquintan ($1 + 1/5$), sesquisixtan ($1 + 1/6$), and sesquiseptan ($1 + 1/7$) are found, all of which are sesquipartials.

Inversely, the relationships of the smaller terms to the larger terms are called sub-sesquipartial, for in the same way that the relationship of 3 to 2 is called sesquialter, by analogy, the relationship of 2 to 3 is called sub-sesquialter. In the same way again the relationship of 3 to 4 is called sub-sesquitertian.

Among the multiple relationships, the first and smallest is the double, next comes the triple, then the quadruple, and so forth increasing indefinitely.

Among the sesquipartial relationships, the first and largest is the sesquialter relationship($1 + 1/2$), because the fraction $1/2$ is the first, the largest and the one which most closely approaches the whole; Then comes the sesquitertian relationship ($1 + 1/3$), then the sesquiquartan relationship ($1 + 1/4$) and so forth indefinitely, always proceeding by diminishing.

THE EPIMER RELATIONSHIP

XXV. A relationship is called *epimer* when the larger term contains the smaller one time plus several other parts of it, either similar parts or different parts; similar as two-thirds, two-fifths, etc. Thus the number 5 contains 3 plus two-thirds of 3; the number 7 contains 5 plus two-fifths of 5; the number 8 contains 5 and three-fifths of 5; and so on. The parts are different when the largest term contains the smallest and in addition, its half and its third, as in the relationship of 11 to 6, or its half and its quarter, as in the relationship of 7 to 4, or again, by Zeus [23], the third and the quarter, as in the relationship of 19 to 12 [24].

Similarly, other epimer relationships can be recognized which are greater than unity by two, three or a greater number of parts,

[23] This exclamation "by Zeus" actually exists in the ancient Greek text as met, also in a previous page. The French translator does not mention it in French. Perhaps he thought including exclamations would be beyond the scope of his work. The truth is that a statement followed or preceeded by this exclamation "by Zeus" was the ultimate, something beyond any doubt. It is not a colloquialism as I assume Mr. Dupuis inferred and consequently censored it out of his translation. (Toulis)

[24] In fact we have
$$11/6 = 1 + 5/6 = 1 + 3/6 + 2/6 = 1 + 1/2 + 1/3$$
$$7/4 = 1 + 3/4 = 1 + 2/4 + 1/4 = 1 + 1/2 + 1/4$$
$$19/12 = 1 + 7/12 = 1 + 4/12 + 3/12 = 1 + 1/3 + 1/4$$

whether these parts be similar or not. Inversely the hypepimer relationship, is the one which is obtained by taking, in the preceding relationship, the ratio of the smaller term to the larger.

THE MULTISUPERPARTIAL AND POLYEPIMER RELATIONSHIPS

XXVI. The relationship is called multisuperpartial or multisesquipartial when the larger term contains the smaller two or more times plus a part of this smaller term. 7 contains in this way, 2 times 3 and in addition, a third of 3. Also it is said that the relationship of 7 to 3 is bisesquitertian. Likewise 9 contains 2 times 4 and the fourth of 4 in addition; the relationship of 9 to 4 is bisesquiquartan. Again likewise, 10 contains 3 times 3 along with the third of 3, and the relationship is called trisesquitertian.

Other multisuperpartial relationships are recognized in the same manner. They occur in every case where, of the two proposed numbers, the smaller does not measure the larger exactly, but when the larger gives a remainder which is at the same time a remainder of the smaller. Thus the relationship 26 to 8 is multisuperpartial because 3 times 8 does not give 26 completely, but comes to 24 rather than 26, and there is a remainder of 2 which is a quarter of 8.

XXVII. A relationship is called polyepimer when the larger term contains the smaller two times or more, along with two or several parts of the latter, whether they be similar or different. Thus 8 contains 2 times 3 and in excess, two-thirds of 3, and the relationship is called double with two-thirds in excess ($2 + {}^2/_3$); likewise the relationship of 11 to 3 is triple with two-thirds in excess ($3 + {}^2/_3$); the relationship of 11 to 4 is double with three-quarters in excess (${}^{11}/_4 = 2 + {}^3/_4 = 2 + {}^1/_2 + {}^1/_4$).

It is easy to find many other polyepimer relationships, and this takes place each time that the smaller number does not exactly measure the larger, but there is a remainder formed of several parts of the smaller number, as in the relationship of 14 to 3, since 3 does not exactly measure 14, but 4 times 3 are 12, and of 14 there remains 2 which is two parts of three and which is called two-thirds. To the polyepimer relationship is opposed the hypo-polyepimer relationship (the inverse relationship).

XXVIII. A ratio of number to number is what takes place when the larger has none of the relationships we have spoken of with the

smaller. As will be shown, it is a relationship of number to number, reduced to its smallest terms, which measures the leimma, the relationship which is 256 to 243.[25] It is evident that the ratio of the smaller numbers to the larger numbers is in the inverse. It borrows its name from the first relationships, as has been shown.

THE FOUNDATION OF RELATIONSHIPS

XXIX. Of all the relationships which we have discussed in detail, those which are expressed in the smallest numbers and primary to each other are called "firsts" or the basic of all the relationships of a similar (or equal) species. Thus the first and the basic of the double relationships is the relationship of 2 to 1, for after this the double relationships are expressed in larger and composite numbers, like the relationships of 4 to 2, and 6 to 3 and so forth indefinitely.

In the same way the first and the basic of the triple relationships is the relationship 3 to 1, and the triple relationships expressed in larger and composite numbers go on to infinity. It is the same with other multiple relationships and superpartial relationships, the first and the basic of the sesquialter relationships is $3/2$; for the sesquitertian relationship it is $4/3$, for the sesquiquarten relationship it is $5/4$. There is an infinite number of equivalent relationships expressed in larger, composite terms. The same observations can be made of other relationships.

THE DIFFERENCE BETWEEN THE INTERVAL AND THE RELATIONSHIP

XXX. The interval and the relationship differ in that the interval is contained between homogenious and unequal terms, while the relationship simply links homogenious terms to one another. This is why there is no interval between equal terms, but there is a relationship between them which is that of equality. Between unequal terms, the interval of one to the other is unique and identical, while the relationship is varied and inverse from one term to the other: thus 2 to 1 and 1 to 2 have only a single and indentical interval, but there are two different relationships, the relationship of 2 to 1

[25] The relationship of 256 to 243 is epimer, since we have: $256/243 = 1 + 13/243 = 1 + 9/243 + 3/243 + 1/243 = 1 + 1/27 + 1/81 + 1/243$, so that the larger term contains the smaller one time and in addition, several different parts of it. Cf. II, XXV.

being double, whereas the relationship of 1 to 2 is a half.

Eratosthenes, in the *Platonist* also said that the interval and the relationship are not the same thing, because the relationship is a specific connection of two magnitudes to each other, and it exists between different and also not different things, as when it is said that the perceptible is to the intelligible in the same relationship as opinion is to scientific knowledge, or that the intelligible differs from the known in the same relationship as the perceptible differs from opinion, while these things differ only by a single interval, whether of size, of quality, of position or in any other way. By this it is evident that the relationship is a different thing than the interval, because the half and the double do not form the same relationship while they do form the same interval.

XXXI The proportion is a similarity or identity of several relationships, that is to say a similarity of ratios in several terms, which takes place when the relationship of the first term to the second is equal to the relationship of the second to the third or to the relationship to two other terms. The first proportion is called continuous and the second is called discontinuous. At least three terms are necessary for a continuous proportion, and the discontinuous requires at least four terms.

After the proportion formed of equal terms, the three smallest terms, 4, 2, 1 in double ratio form a continuous proportion, because 4 is to 2 as 2 is to 1; and the numbers 6, 3, 4, 2 form a discontinuous proportion, since 6 is to 3 as 4 is to 2. The same thing is observed with other multiple relationships, and the continuous proportion is in a certain way a four term proportion, because of the repetition of the mean term. The explanation is the same when the relationships are sesquipartial: thus the numbers are 9, 6, 4, in sesquialter relationship ($1 + {}^1/_2$), form a continous proportion, and the terms 9, 6, 15, 10, form a discontinuous proportion. The same would be found of proportions having other relationships.

Eratosthenes says that the ratio is the principle which gives birth to the proportion and it is also the primary cause of the creation of all orderly things. Every proportion is indeed composed of ratios and the principle of the ratio is equality. This is evident: for each species there is a certain element or principle which belongs to it, into which all the others are resolved, while it itself does not resolve into any of the others. However this principle is necessarily

unresolvable and indivisible, for anything that can be broken down and divided is called a bond and not an element.

The elements of substance are therefore indivisible according to substance; those of quality are so according to quality; those of quantity are so according to quantity. And each thing is indivisible and whole as it is an element of a composed or mixed thing. Thus the element of quantity is the unit; that of magnitude is the point; that of relationship and of proportion is equality. For unity cannot be divided in quantity, nor the point in size, nor equality in multiple relationships. Number is born out of the unit, the line out of the point, relationship and proportion out of equality; but not all in the same manner, because unity multiplied by itself does not create like the other numbers: one times one is one, while by addition it increases to infinity.

As for the point, it is neither by multiplication nor by addition that it forms the line, but by a continuous movement, in the same way that the line forms a surface and the surface a solid. Similarly the ratio of equality does not increase by addition, for if one adds several equal relationships in order, the ratio of the sum again gives an equality. Thus the point is not a part of the line, nor is equality a part of the relationship. In every case unity makes up part of the number, for it can increase only by the repetition of itself. The cause of what we have just said is that equality has no interval, just as the point has no size.

Plato seems to believe that the connecting element of mathematics is unique and that it consists in proportion. He says, indeed in the *Epinomis*[26] that it is necessary that every figure, every combination of numbers, every harmonic group, and every astronomical revolution manifest the uniqueness of proportion, to him who learns according to the true method. And this uniqueness will be apparent to whomever comprehends correctly what we teach: he will understand that a single bond naturally unites all things.

XXXII. A mean number differs from a mean proportion.[27] For if a number is a proportional mean between two others, it is a term contained between them; but if a term is contained between two others it is not necessarily a proportional mean between these numbers. It can in fact happen that a number contained between two extremes might not be in proportion with them, like 2 which is con-

[26] *Epinomis,* 991 E — 992A

[27] The Greek word Μεσότης means 'median' (or médiate in French) and here it can be manifested as a mean number. (Toulis)

tained between 1 and 3, and 2, 3, 4, which are contained between 1 and 10, because 10 cannot be reached without passing through 2, 3, 4 and yet none of these numbers is in proportion with the extremes, since the relationship of 1 to 2 is not equal to that of 2 to 3, and likewise also the relationship of 1 to 2, 3 or 4 is not equal to that of 2, 3 or 4 to 10. The proportional means between two numbers are, on the contrary, contained between these numbers: thus in the proportion 1, 2, 4, whose ratio is double, the proportional mean 2 is contained between 1 and 4.

ON PROPORTIONS (BETWEEN THREE NUMBERS)

XXXIII. Thrasyllus takes three principle proportions between three numbers into account: the arithmetic proportion, the geometric proportion, and the harmonic proportion. The arithmetic proportion is that in which the mean term surpasses one extreme term by the same amount as it is surpassed by the other extreme, such as the proportion 1, 2, 3; the geometric proportion is that in which the mean term contains one extreme, such as the proportion 3, 6, 12. The harmonic proportion between three numbers is that in which the mean number surpasses one extreme number and is surpassed by the other, by the same fraction of the extreme numbers, like the third, and the fourth, such is the proportion of the numbers 6, 8, 12.

Thus each of the relationships can be considered in this way. 12 is the double of 6, 18 is its triple, 24 its quadruple; 9 is $3/2$ of it and 8 is $4/3$ of it; 9 is $9/8$ of 8; 12 is $4/3$ of 9, and $3/2$ of 8 (the double of 6); 18 is the double of 9 and 27 is the $3/2$ of 18: $8/6$ gives the consonance of the fourth, $9/6$ the consonance of the fifth and $12/6$ the consonance of the octave; $18/6$ gives and octave and a fifth, for 12 being the double of 6 forms the consonance of the octave and 18 being $3/2$ of 12 is the consonance of the fifth: we have the relative numbers 6, 12, 18.[28] $24/6$ gives the consonance of the double octave; $9/8$ gives the tone and $12/9$ the fourth; $12/8$ gives the fifth and $18/9$ the octave. The ratio $27/18$ gives the fifth.

The octave $12/6$ is composed of the fifth $9/6$ and the fourth $12/9$, or again the fifth $12/8$ and the fourth $8/6$.[29] The octave $18/9$ is composed

[28] $\frac{18}{6} = \frac{12}{6} \times \frac{18}{12} = 3$.

[29] $\frac{12}{6} = \frac{8}{6} \times \frac{12}{8} = \frac{12}{9} \times \frac{9}{6} = 2$.

of the fifth $^{18}/_{12}$ and the fourth $^{12}/_{9}$;[30] the ratio $^{24}/_{12}$ for the octave is composed of the ratio $^{24}/_{18}$ for the fourth and the ratio $^{18}/_{12}$ for the fifth.[31] Finally the ratio $^{9}/_{6}$ which is a fifth is composed of a tone $^{9}/_{8}$ and a fourth $^{8}/_{6}$[32] and the ratio $^{12}/_{8}$ is thus a fifth, composed of a fourth $^{12}/_{9}$ and a tone $^{9}/_{8}$.[33]

XXXIV. The leimma is in the relationship of the number 256 to the number 243. This is how the relationship is found: we take the sesquioctave relationship two times (the two terms of the first are multiplied by 9, the two terms of the second by 8), and triple the result. Then we join the sesquitertian relationship to it. The sesquioctave relationship being that of 9 to 8, we form with these two numbers two other sesquioctave relationships in the following manner: 9 x 9 = 81; 9 x 8 = 72; and 8 x 8 = 64; 81 is $^{9}/_{8}$ths of 72 and 72 is $^{9}/_{8}$ths of 64. If we triple these numbers, we have 81 x 3 = 243; 72 x 3 = 216 and 64 x 3 = 192. $^{4}/_{3}$rds of 192 is 256. This number compared with 243 gives the relationship of the leimma which is less than 1 + $^{1}/_{18}$th.[34]

256 243 216 192

leimma tone tone

fourth

THE DIVISION OF THE CANON

XXXV. The division of the canon is made according to the quaternary[35] of the decad which is composed of the numbers 1, 2, 3, 4 and which embraces the sesquitertian, sesquialter, double, triple and quadruple ratios (that is $^{4}/_{3}$, $^{3}/_{2}$, 2, 3, and 4).

This is how Thrasyllus divided this canon. Taking half of the string he obtains the mese consonance of the octave which is in

[30] $^{18}/_{9} = {}^{18}/_{12} \times {}^{12}/_{9}$

[31] $^{24}/_{12} = {}^{24}/_{18} \times {}^{18}/_{12}$

[32] $^{9}/_{6} = {}^{9}/_{8} \times {}^{8}/_{6}$

[33] $^{12}/_{8} = {}^{12}/_{9} \times {}^{9}/_{8}$

[34] The leimma is *less* than 1 + $^{1}/_{18}$th. The fraction $^{13}/_{243}$ is actually less than $^{1}/_{18}$, therefore 1 + $^{13}/_{243}$ or $^{256}/_{243}$ is less than 1 + $^{1}/_{18}$th.

[35] The "tetraktys" of the Pythagoreans — The number 10 or its parts 1 + 2 + 3 + 4 = 10. (Toulis)

double ratio, the tension being double for the higher pitched sounds, in the inverse direction of the movement. The inversion is such that, when the total length of the string is diminished by the canon, the pitch is proportionately raised, and that when the length is increased the pitch is proportionately lowered, because the half-length of the proslambanomenos, which is the mese with respect to the total string, has a double tension toward the higher pitches, and the total string which is double has a half-tension on the side of the low sounds.

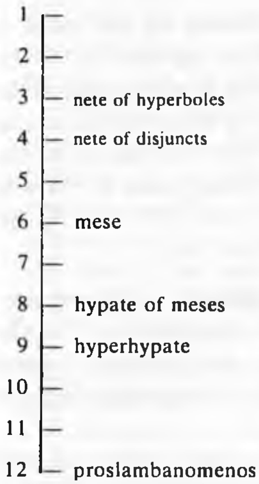
The division of the string into three gives the hypate of the meses and the nete of the disjuncts, the nete of the disjuncts being a fifth of the mese since the divisions are in the relationship of 2 to 3, and it is to the hypate (of the meses) in the relationship of an octave, since the divisions are as 1 to 2. The nete of the disjuncts gives with the proslambanomenos the consonance of the octave and a fifth, because from the proslambanomenos to mese there is one octave and, the intervals being prolonged up to the nete of the disjuncts, there is a fifth from this to the mese.

From the mese to the hypate (of the meses) there is a fourth, and from the mese to the proslambanomenos there is an octave, the hypate of the meses giving the fifth through the relationship to the proslambanomenos. The same distance of an octave is obtained by adding hypate (of the meses) to the mese, which is a fourth, to the interval from mese to the nete of the disjuncts, which is a fifth. The numbers of movements (that is to say vibrations) varies in the inverse direction from the division of the lengths (that is in the inverse direction of the length of the vibrating part).

By dividing the string in fourths, the diatone of the hypates, also called the hyperhypate, and the nete of the hyperboles are obtained. The nete of the hyperbole is to the nete of the disjuncts in the relationship of the fourth, to the mese in the relationship of the octave, to the hypate (of the meses) in the relationship of the octave and a fourth, to the hyperhypate in the relationship of the octave and a fifth, and to the proslambanomenos in the relationship of the double octave, in going toward the low tones.

The hyperhypate is to the proslambanomenos in the relationship of the fourth, going toward the low tones, and to the mese in the relationship of the fifth, going toward the high tones; it is one tone below the hypate (of the meses) and the interval from hyperhypate to the last cord (the proslambanomenos) is equal to

the interval of the fourth from the nete of the disjuncts to the nete of the hyperboles; and here again the number of movements (vibrations) is in the inverse direction to the size of the divisions.[36]



All of this will be made evident through the numbers, because if one divides the length of the canon into twelve appropriate parts, the mese will be given by each half of the total string. The hypate of meses will be given by blocking off four parts at the beginning of the canon and the nete of disjuncts in taking four parts at the other end of the canon, in such a way that there will be four parts between them. The hyperhypate will be given by blocking off three parts at the beginning; it is spaced by one division from the hypate (of meses). The hyperboles (nete of the hyperboles) is obtained by taking three parts of the string; it is spaced by one division from the disjunct (nete of the disjuncts).

Between the hyperhypate and the nete of the hyperboles, there are 6 divisions, three above the mese and three below it; and this completes the partitioning. In fact, from the beginning of the canon to the hyperhypate, three parts of the canon can be counted, from there to the hypate of the meses, one part, and from the latter to the mese, two parts. From the mese to the nete of the disjuncts, there are two parts, from there to the hyperbole one part, and finally from the latter to the end of the canon, three parts. There is, then, a total of 12 divisions.

The ratio of the nete of the disjuncts to the nete of hyperboles is $^4/_3$, which is the sesquitertian relationship which gives the consonance of the fourth. The relationship of the mese to the nete of the hyperbole is $^6/_3 = 2$ which is the consonance of the octave. The ratio of the hypate of the meses to the same nete $^8/_3$, the consonance of the octave and a fourth. The ratio of the hyperhypate to the nete is $^9/_3 = 3$, the consonance of the octave and a fifth, and the relationship of the proslambanomenos to it is $^{12}/_3 = 4$, the consonance of the double octave. The ratio of the mese to the nete of disjuncts equals $^6/_4 = {}^3/_2$, which is the sesquialter relationship, the consonance of the fifth. The interval of the hypate (of meses) to the nete

[36] See note XII.

of disjuncts equals $^8/_4 = 2$, the octave. That of the hyperhypate to the same nete equals $^9/_4$, which is the double fifth (fifth of the fifth). For the complete proslambanomenos, the relationship is $^{12}/_4 = 3$, the consonance of the octave and a fifth.

The relationship of the hypate of the meses to the mese is $^8/_6 = ^4/_3$, the fourth. That of the hyperhypate to the mese is $^9/_6 = ^3/_2$ and gives the fifth. That of the complete proslambanomenos to the mese is $^{12}/_6 = 2$, the octave. The hyperhypate is to the hypate of meses as 9 is to 8, the ratio of the tone. The relationship of the whole proslambanomenos to the hypate of the meses is $^{12}/_8 = ^3/_2$ (the fifth). The same string is to the hyperhypate as 12 is to 9, this relationship equals $^4/_3$, the consonance of the fourth.

XXXVI. The numbers of vibrations are subject to inverse proportion, since the tone, the ratio of which is sesquioctave ($^9/_8$), the consonance of the fourth the ratio of which is sesquitertian ($^4/_3$) and the consonance of the fifth, the ratio of which is sesquialter ($^3/_2$), are found condensed in the canon.

The $^3/_2$ ratio of the fifth is greater than the ratio $^4/_3$ by one tone which is equal to $^9/_8$: let us take for example the number 6 which is divisible by 2 and by 3: $^4/_3$ of 6 equal 8 and $^3/_2$ of 6 equal 9, and 9 is $^9/_8$ of 8. We then have 6, 8, 9 and the excess of the interval $^3/_2$ over the interval $^4/_3$ is $^9/_8$. But the interval $^4/_3$ of the fourth is composed of two times $^9/_8$ and a leimma, the intervals must then be filled by tones and leimmas. This insertion begins at the nete of the hyperboles, indeed if we extend the latter an eighth part of its length, we will have the diatone of the hyperboles, which is lower by one tone.

If we prolong the diatone an eighth part of its length, we will have the trite of the hyperboles, which is lower by one tone than the diatone.* The remainder of the interval up to the nete of the disjuncts will be the leimma, complement of the consonance of the fourth through relationship to the nete of the hyperboles. If on the contrary, we diminish by a ninth the length of the nete of the disjuncts, we will have the chromatic of the hyperboles, which is a tone higher than the nete of the disjuncts; this augmented by one eighth will give the paranete of disjuncts, which is also called the diatone, and nete of conjuncts, and which is lower by one tone than the nete of disjuncts.

Then if we prolong the nete of the conjuncts by one eighth of its length, we will have the trite of the disjuncts, one tone lower, and which is the same as the diatone of the conjuncts. And the re-

* See the Perfect System on page 148.

mainder of the interval up to the paramese will be the leimma. If we prolong the paramese by one eighth, we will have the mese, lower by one tone, and which completes the octave. If we diminish the mese in the same manner (by retracting a ninth of its length) we will have the paramese or chromatic of the conjuncts, higher by one tone than the mese; and in subtracting its ninth part from this, we will have the chromatic of the disjuncts.

The mese, increased by an eighth, will give the diatone of the meses, lower by one tone than the mese; the diatone of the meses, increased by an eighth, gives the parhypate of the meses, lower by one tone, and from there to the hypate of the meses there remains a leimma for the complement of the consonance of the fourth with the mese. If one subtracts one ninth from the hypate of the meses, one obtains the chromatic of the meses, which is higher by one tone, and if on the other hand, one increases it by one eighth, the hyperhypate is obtained, which, increased by one eighth gives the parhypate of the hypates.

Reciprocally, if the length of the proslambanomenos is divided into 9 parts, and one of these parts is taken away in the inverse of what we have done for the high tones, we will have the hypate of the hypates, higher by one tone than the proslambanomenos and terminating the tetracord of the hypates by the relationship of the leimma which it has with the parhypate. In this way the whole immutable system of the diatonic type and the chromatic type are completed.

The enharmonic system is derived from the diatonic system by removing the diatones that are heard two times in each octave and by dividing the half-tones in two.

We will find the numerical results by beginning with the nete of the hyperboles which we will suppose to be of 384 parts, of which we take successively $^9/_8$ths and the other fractions we have indicated. The proslambanomenos will have the value of 10368.[37] It is unnecessary to develop this in detail because anyone who has understood the foregoing will find it easy to calculate.

Such is the division of the canon given by Thrasyllus. When we put forth the elements of astronomy we will show how all of this applies to the system of the worlds. Let us now go on to the explanation of the other means and the mean numbers since, as we have said, every mean is a mean number, but every mean number is not a mean. It is then in so far as the mean is a mean number that it

[37] See note XIII.

is necessary to understand the following, concerning means and mean numbers.

THE QUATERNARY (OR TETRAKTYS) AND THE DECAD

XXXVII. Since, as we have shown, all the relationships of the consonances are found in the quaternary of the decad, it is of these numbers that we have to speak. The decad in, fact, constitutes the quaternary, since the sum of the numbers 1, 2, 3, 4 is 10. But these numbers contain the consonance of the fourth in the sesquitertian ($^4/_3$) relationship, that of the fifth in sesquialter ($^3/_2$) relationship, that of the octave in the double ratio and that of the double octave in the quadruple ratio; and in this way the immutable diagram is completed.

HOW MANY QUATERNARIES (TETRAKTYS) ARE THERE?

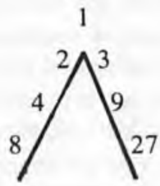
XXXVIII. The importance of the quaternary obtained by addition (that is to say 1, + 2, + 3, + 4) is great in music because all the consonances are found in it. But it is not only for this reason that all Pythagoreans hold it in highest esteem: it is also because it seems to outline the entire nature of the universe. It is for this reason that the formula of their oath was: "I swear by the one who has bestowed the tetraktys to the coming generations, source of eternal nature, into our souls". [38] The one who bestowed it was Pythagoras, and it has been said that the tetraktys appears indeed to have been discovered by him.

The first quaternary is the one of which we've just spoken: it is formed by addition of the first four numbers.

The second is formed by multiplication, of even and odd numbers, starting from unity. Of these numbers, unity is the first because, as we have said, it is the principle of all the even numbers, the odd numbers and of all the odd-even numbers, and its essence is simple. Next comes three numbers from the odd as well as the even series. They allow for the unification of odd and even because numbers are not only odd or even. For this reason, in multiplication,

[38] Cf. *Golden Verses of Pythagoras*, 47-48; Macrobius, *Commentary on the Dream of Scipio*, I, 6: *Theologumena Arithmeticae* IV, p. 18, edited by Ast. Iamblicus, *Life of Pythagoras*, para. XXVIII and XXIX; The Emporor Julien, *Against the Christians* (philosophy of the cynics), para. II; Plutarch, *Opinions of the Philosophers*, I, III; etc.

two quaternaries are taken, one even, the other odd; the even in double ratio, the first of the even numbers being 2 which comes from unity doubled; the odd in triple ratio, the first of the odd numbers being the number 3 which arises from unity being tripled, so that unity is odd and even simultaneously and belongs to both. The second number in the even and double (series) is 2 and in the odd and triple is 3. The third of the order of even numbers is 4, and in the odd series, 9. The fourth among the even numbers is 8, and among the odd numbers, 27.



The ratios of the most perfect consonances are found in these numbers; even the tone is included. However unity contains the principle of ratio, of limit and of point. The second numbers, 2 and 3 have the side ratio, being prime, incomposite numbers, and measured only by the unit, and are consequently linear numbers. The third terms, 4 and 9, have the power of the squared surface, being equally equal (that is to say square numbers). The fourth terms, 8 and 27, have the power of the cubic solid, being equally equal equally (that is to say, cubic numbers). In this way, by virtue of the numbers from this tetraktys, growth procedes from the limit and the point up to the solid. In fact, after the limit and the point comes the side, then the surface and finally the solid. It is with these numbers that Plato, in the *Timaeus*, constitutes the soul.[39] The last of these seven numbers is equal to the sum of all the preceding, as we have $1 + 2 + 3 + 4 + 8 + 9 = 27$.

There are then two quaternaries of numbers, one which is made by addition, the other by multiplication; and these quaternaries encompass the musical, geometric and arithmetic ratios of which the harmony of the universe is composed.

The third quaternary is that which, following the same proportion, embraces the nature of all magnitudes, for the place taken by unity, in the preceding quaternary, is that of the point in this one, and that of the numbers 2 and 3, having lateral (or linear)

[39] Plato, the *Timaeus*, 36 bc.

power, is here that of the *line*, through its double form, straight or circular, the straight line corresponding to the even number because it terminates at two points (the line and circle are given as examples here), and the circular to the odd, because it is composed of a single line without terminus.

And what, in the preceding quaternary, are the numbers 4 and 9, having the power of the surface, the two types of surface, the planar and the curved, are so (surface) in this one: Finally, what, in the preceding are the numbers 8 and 27, which have the power of the cube and of which one is even and the other odd, is constituted by the solid in this one. There are two kinds of solids, one with a curved surface, like the sphere or the cylinder, the other with a plane surface, such as the cube and the pyramid. This is the third tetraktys then, the one having the property of constituting any magnitude, through the point, the line, the surface and the solid.

The fourth quaternary is that of the simple bodies, fire, air, water and earth, and it offers the same proportion as the quaternary of numbers. The place occupied by unity in the quaternary of numbers is taken by fire in this one, air corresponds to the number 2, water to the number 3, earth to the number 4; such is indeed the nature of the elements according to their fineness or density, in such a way that fire is to air as 1 is to 2, to water as 1 is to 3, and to earth as 1 is to 4. The other relationships are also equal (that is to say, that air is to water as 2 is to 3, and so forth for the others).

The fifth quaternary is that of the shapes of simple bodies, for the pyramid is the figure of fire, the octahedron the figure of air, the icosahedron [40] the figure of water and the cube the figure of earth.

The sixth is that of the created things, the seed being analogous to unity and to the point. A growth in length is analogous to the number 2 and to the line, and a growth in width is analogous to the number 3 and to the surface, and finally a growth in thickness is analogous to the number 4 and to the solid.

The seventh quaternary is that of societies. Man is principle and is thus, unity. The family corresponds to the number 2, the village to the number 3 and the city to the number 4; for these are the elements which compose the nation.

All of these quaternaries are material and perceptible.

The eighth contains faculties by which we are able to form judgment on the preceding, and which are its intellectual part,

[40] A solid having 20 surfaces. See: "Mathematics of the Cosmic Mind," by L. Gordon Plummer, 1972.

namely: thought, science, opinion and feeling. And certainly thought, in its essence, must be assimilated to unity;science is the number 2, because it is the science of all things, opinion is like the number 3, because it is something between science and ignorance; and finally feeling is like the number 4 because it is quadruple, the sense of touch being common to all, all the senses being motivated through contact.

The ninth quaternary is that which composes the living things, body and soul, the soul having three parts, the rational, the emotional and the willful; the fourth part is the body in which the soul resides.

The tenth quaternary is that of the seasons of the year, through the succession of which all things take birth, that is, spring, summer, autumn and winter.

The eleventh is that of the ages: childhood, adolescence, maturity and old age.

There are thus eleven quaternaries. The first is that of the numbers which are formed by addition, the second is that of the numbers formed by multiplication, the fourth is that of magnitudes, the fifth is that of simple bodies, the sixth is that of created things, the seventh is that of societies, the eighth is that of the faculties of judgment, the ninth is that of the living thing, the tenth is that of the seasons, and the eleventh is that of the ages. They are proportional to one another, since what is unity in the first and the second quaternary, the point is in the third, fire in the fourth, the pyramid in the fifth, the seed in the sixth, man in the seventh, thought in the eighth, and so forth with the others following the same proportion.

Thus the first quaternary is 1, 2, 3, 4. The second is unity, the side, the square, the cube. The third is the point, the line, the surface, the solid. The fourth is fire, air, water, earth. The fifth is the pyramid, the octahedron, the icosahedron, the cube. The sixth is the seed, the length, the width, the height. The seventh is man, the family, the village, the city. The eighth is thought, science, opinion, sense. The ninth is the rational, the emotional and the willful parts of the soul, and the body. The tenth is spring, summer, autumn, winter. The eleventh is childhood, adolescence, maturity and old age. And the perfect world which results from these quaternaries is geometrically, harmonically and arithmetically arranged, containing in power the entire nature of number, every magnitude and every body, whether simple or composite. It is perfect because

everything is part of it, and it is itself a part of nothing else. This is why the Pythagoreans used the oath whose formula we have reported, and through which all things are assimilated to number.

THE DECAD

XXXIX. The Pythagoreans were no less wise in bringing all numbers back to the decad, since we do not count any number beyond ten: in going beyond ten we go back to the numbers 1, 2, 3 and so on. The decad is,however,found in the quaternary, since the sum of the four numbers 1, 2, 3, 4 is equal to 10, from which it follows that the strongest numbers can be considered as having their ratio in the quaternary.

PROPERTIES OF THE NUMBERS CONTAINED IN THE DECAD (TETRAKTYS)

XL. Unity is the principle of all things and the most dominant of all that is: all things emanate from it and it emanates from nothing. It is indivisible and it is everything in power. It is immutable and never departs from its own nature through multiplication ($1 \times 1 = 1$). Everything that is intelligible and not yet created exists in it: the nature of ideas, God himself, the soul, the beautiful and the good, and every intelligible essence, such as beauty itself, justice itself, equality itself, for we conceive each of these things as being one and as existing in itself.

XLI. The first increase, the first change from unity is made by the doubling of unity which becomes 2, in which are seen matter and all that is perceptible, the generation of motion, multiplication and addition, composition and the relationship of one thing to another.

XLII. The number 2 added to unity produces 3, which is the first number having a beginning, a middle and an end. This is why this number is the first to which the name *multitude* [41] applies, for all the numbers less than this are not called multitude (or many) but one or one and other; while three is called multitude. We make *three* libations to show that we ask *everything* which is good. We call thrice woeful those who are at the limit of misfortune, and thrice favored those who are at the limit of success.

[41] Cf. Plutarch, *Opinions of the Philosophers*, I, III, 23: "the number three expresses the multitude." See also, *On Isis and Osiris*, 36.

The ternary number also represents the first nature of the plane because it is in a sense its image, the first form of the plane being the triangle. It is for this reason that there are three types of triangles, the equilateral, the isosceles and the scalene, and also that there are three types of angles, the right angle, whose property it is to be unique, well defined and composed of the equal and the similar, and which causes all right angles to be equal to one another, being in the middle between the acute and the obtuse angles, larger than one and smaller than the other. As for all the other angles, they are infinite in number, and indetermined, since they are either greater or smaller. The number 3 added to unity and to 2 gives 6 which is the first perfect number, that is to say equal to the sum of its aliquot parts. This perfect number, added to the first square number, 4, gives the decad.

XLIII. The number 4 is the image of the solid, and it is the first square number among the even numbers; it completes all the consonances, as we have shown.[42]

XLIV. The number 5 is the mean term of (two numbers whose sum is) the decad, because if, by the addition of any two numbers, one obtains ten, the means of these numbers will be 5 according to arithmetic proportion. Thus, for example, if you add 9 and 1, 8 and 2, 7 and 3, 6 and 4, the sum will always be 10 and the mean term in arithmetic proportion will be 5. This is shown in the diagram in which any addition of two opposite numbers gives 10, the proportional arithmetic mean being 5, which is greater than one of the extremes and less than the other, by the same difference.

1	4	7
2	5	8
3	6	9

This number is also the first which embraces the two types of numbers, the odd and the even, that is, 2 and 3, for unity is not a number.

XLV. The number six is a perfect number because it is equal to the sum of its aliquot parts, as has been shown. This is why it is

[42] The number four is the image of the solid because the most elementary of the solids is the triangular pyramid which has 4 faces and 4 apexes. And it completes the consonances which are $4/3$, $3/2$, 2, 3, and 4, that is, the fourth, the fifth, the octave, the fifth of the octave and the *double octave*. Cf. above, II, VI.

called that of marriage, because the task of marriage produces children similar to their parents.[43] The harmonic mean is composed according to this primary number, since in taking its four-thirds, 8, and its double, 12, one arrives at the harmonic proportion of the numbers 6, 8, 12. Eight is greater than one of the extremes, 6, and less than the other extreme, 12, by the same fraction of the extremes, which is one-third of the extremes. Six also gives the arithmetic mean by taking 9, which is $^3/_2$ rds of it, and 12 which is its double, since 9 is greater than one of the extremes and less than the other by the same quantity, 3. Finally, six produces the geometric proportion when, being placed in the middle, half of it, 3, is placed on one side, and its double, 12, is placed on the other, giving the geometric proportion of the numbers 3, 6, 12, since 6 then contains one of the extremes, and is contained by the other in the same relationship, i.e. 2.

XLVI. Another number of the decad, the number seven, is endowed with a remarkable property: it is the only one which does not give birth to any number contained in the decad and which is not born out of any of them, which fact moved the Pythagoreans to give it the name Athena, because this goddess was not born out of a mother and gave birth to none. This number did not arise from any union, and does not unite with anything. Among the numbers contained in the decad, some create and some are created, for example 4 multiplied by 2 creates 8, and is created by 2. Others are created but do not create, like 6 which is the product of 2 by 3, but which does not create any of the numbers in the decad. Others create but are not created, such as 3 and 5 which are not created by any combination of numbers, but which create: 3 produces 9, and multiplied by 2 produces 6, and 5 multiplied by 2 produces 10.

Seven is the only number which, multiplied by another number, creates none of the numbers in the decad, and which is not produced by the multiplication of any number. Plato in the *Timaeus*,[44] following nature, endows the soul with 7 numbers. . . . Day and night, said Posidonius, have the nature of the even and the odd. . . . The month is composed of four weeks (*four* times *seven* days); in the first week the moon appears divided in two; in the second it becomes full, in the third it is again divided, and in the fourth, it returns to meet the sun in order to begin a new month and to increase during the following week.

[43] See note XIV.

[44] *Timeaus* 35b.

It is in seven weeks that the foetus appears to arrive at its perfection, as Empedocles insinuates in his *Expiations*. Some think that the male foetus requires five weeks for its perfection. It is also in the seventh month that the foetus can be born living. Children develop teeth starting from the seventh month after birth, and fully produce their teeth in seven years, the semen and puberty make their appearance at the age of fourteen, and often it is in the third period, i.e. at the age of twenty-one, that the beard begins to grow. It is then also that a man acquires his full height, but it is only in the fourth period, i. e. at twenty eight, that he acquires his stoutness.

Seven days are needed to diagnose illness, and in all periodic fevers, even in three and four-day fevers, the seventh day is always the most serious. From one solstice of the sun to the other there are seven months, and the planets are seven in number. Similarly seven months are counted from one equinox to the other. [45] The head has seven orifices. There are seven viscera, the tongue, the heart, the lungs, the liver, the spleen, and the two kidneys. Herophilus says that the intestine of man is 28 cubits long, that is to say, four times seven cubits. Finally, in most straits, the ebb-tide reverses direction seven times per day.[46]

XLVII. The number eight which is the first cube is composed of unity and the septenary. Some people say that there are eight principle gods in the universe, and this is also found in the oaths of Orpheus:

> By the creators of things forever immortal:
> fire and water, earth and heaven, the moon
> and the sun, the great torch and the black night.

And Evander relates that in Egypt, on a column, is found an inscription dedicated to King Saturn and Queen Rhea:

"To the most ancient of all, the king Osiris, one of the immortal gods, the spirit, the sky and the earth, the night and the day, the father of all that is and all that will be, and to Love, monument to his magnificence and tribute of his life". Timotheus also relates the proverb "eight is all," because the spheres of the world which turn

[45] From one solstice of the sun to the other and from one equinox to the other there are only six months. It is necessary then to understand Theon's thought as follows: starting from one tropic or from one equinox, the sun reaches the other tropic or equinox in the seventh month.

[46] See note XV.

around the earth are eight in number. And, as Eratosthenes also says:

"These eight spheres also harmonise together while making their revolutions around the earth."

XLVIII. The number nine is the first square among the odd numbers: the two first numbers are 2 and 3, one even, the other odd, which give the two first squares of 4 and 9.

XLIX The decad completes the series of numbers, containing in itself the nature of both even and odd, of that which is in motion and that which is still, of good and of evil. Archytas, in his book *On the Decade*, and Philolaus, in his treatise *On Nature*, write at length on this subject.

THE MEDIANS (THE MEAN)

L. Let us return now to proportions and to the means. There are several means : the geometric, the arithmetic, the harmonic, the subcontrary, the fifth and the sixth, to which six others must be added which are subcontrary to them. However, of all of these medians, Adrastus says that the geometric is the only one which is a true proportion, and so it is the first, for all the others have need of it, whereas it has no need of the others, as he then demonstrates. He says that some give the other means the more general name of proportion.

Among the primary proportions themselves, that is to say the geometric proportions, some have rational terms and relationships, such as the proportion 12, 6, 3, whose terms are in double ratio, or any other numerical proportion; others have inexpressible and irrational terms (size, weight, time or other), in double, triple, and in general multiple or sesquipartial ratio. In the geometric mean , the mean term, as we have said, is contained in one extreme and contains the other in the same relationship ($a : b = b : c$). The numerical mean contains and is contained by the same number of extremes. Finally, in the harmonic mean , the mean term is less than one extreme and greater than the other by the same part of the extremes.[47]

LI. Adrastus shows that the ratio of equality is the first in order and that it is an element of all the ratios of which we have pre-

[47] If $a - b = ma$, and $b - c = mc$, then, $a - b : b - c = a : c$

viously spoken and of all the proportions they give, for it is from it that all others take birth, and in it that they are all resolved.

Erathosthenes also says that every ratio increases either by an interval or by terms: but equality has the property of not being susceptible to any interval, and it is evident that it can only grow through the terms. Taking, then, three magnitudes with the proportion found between them, we will combine the terms and show that all mathematics consists in the proportion of certain quantities, and that equality is the principle and the element of proportion.

Eratosthenes says that he will omit the demonstrations but Adrastus shows that "any three terms being given in continuous proportion, if one takes three others formed from these, first an equal one, then another composed of the first and the second, and finally another composed of the first, of two times the second and of the third, these new terms will again be in continuous proportion." [48]

Thus from the proportion whose terms are equal is born a proportion in double ratio, and from the proportion in double ratio is born the proportion in triple ratio, which produces the proportion in quadruple ratio and so forth from the other multiples. We have, for example, in the smallest possible three equal terms, that is to say, in three units, the proportion of equality (1, 1, 1); if we take three other terms in the manner which has been indicated, one formed from the first alone, the other composed of the first and the second, and the last composed of the first, of twice the second and the third, we will have the terms 1, 2, 4, which are in double ratio.

With these numbers, let us form the next one by the same method. The first will be equal to the first, the second will be composed of the first and the second, the third will be composed of the first, of two times the second and of the third, and the terms will be 1, 3, 9, in triple ratio. By the same method, with these numbers, the terms 1, 4, 16 will be formed, which are in quadruple ratio, and with these, the terms 1, 5, 25, in quintuple ratio, and so on to infinity, following the order of the multiples.

[48] If a, b, c, are the three terms given in continuous proportion, we have $b^2 = ac$. The three terms obtained following Adrastus's rule are a, a + b, and a + 2 b + c; The square of the mean term is $a^2 + 2\,ab + b^2$ and the product of the extremes is $a^2 + 2\,ab + ac$. But $b^2 = ac$ by hypothesis, therefore the square of the mean term is equal to the product of the extremes and the three new terms are in continuous proportion.

1	1	1
1	2	4
1	3	9
1	4	16
1	5	25
1	6	36
1	7	49
1	8	64
1	9	81
1	10	100

If now the multiple proportions are arranged inversely, and the terms are added in the same manner, the proportions in sesquipartial ratio will be obtained. The doubles will give in fact the hemiole or sesquialter relationship ($1 + {}^1/_2$), the triples will give the epitrite or sesquitertian relationship ($1 + {}^1/_3$), and the quadruples the sesquiquarten relationship ($1 + {}^1/_4$), and so forth. If, for example, the double ratio proportion in three terms is given, and if the largest term is placed first, 4, 2, 1, with these terms we form the next (proportion) according to the method indicated. We will deduce 4, 6, 9 from them which is a continuous proportion whose relationship is sesquialter.

If we have the same three terms in triple proportion, 9, 3, 1, we will deduce from them in the same manner the three proportional terms in the sesquitertian ratio, 9, 12, 16. With the quadruples, we obtain the terms in sesquiquarten ratio 16, 20, 25 and so forth. We will always have the sesquipartial relationship ($1 + {}^1/n$) corresponding to the multiple (n).[49]

4	2	1	4	6	9
9	3	1	9	12	16
16	4	1	16	20	25
25	5	1	25	30	36
36	6	1	36	42	49
49	7	1	49	56	64
64	8	1	64	72	81
81	9	1	81	90	100

[49] Being in general the continuous proportion n^2, n, 1 whose ratio is n. The new continuous proportion obtained by Adrastus's rule will be formed from the terms n^2, $n^2 + n$, $n^2 + 2n + 1$; the ratio is $1 + 1/n$.

In the same way, the sesquipartial relationships ($1 + 1/n$) give us the epimer relationships

$$(1 + \frac{m}{m+n})$$

and the multisuperpartial relationships ($a + 1/n$); and again the epimer relationships.

$$(1 + \frac{m}{m+n})$$

give us other epimer polyepimer relationships.

$$(a + \frac{m}{m+n}).$$

We can omit most of these relationships as being not very necessary, but we should however consider some of them. With the proportion of the sesquialter ratio ($1 + 1/2$), by beginning with the largest term, one obtains by the method indicated a proportion whose epimer ratio is $1 + 2/3$; thus the proportion 9, 6, 4, by Adrastus's method, gives 9, 15, 25; and, by beginning with the smallest term one obtains the proportion whose multisuperpartial ratio is $2 + 1/2$: one takes 4, 6, 9, and one arrives by the same method, at 4, 10, 25.

And from the proportion in which the relationship is sesquitertian ($1 + 1/3$), by beginning with the largest term, one will derive the proportion in epimer ratio ($1 + 3/4$). We have, in fact, the proportion 16, 12, 9, which gives 16, 28, 49, and by beginning with the smallest term, we will have the proportion in multisuperpartial ratio $2 + 1/3$, in the terms 9, 21, 49. With the proportion in sesquiquartan ratio ($1 + 1/4$), by beginning with the largest term, the proportion in epimer ratio $1 + 4/5$ will be found. The proportion 25, 20, 16, gives indeed 25, 45, 81, and beginning with the smallest term, the proportion in multisuperpartial ratio ($2 + 1/4$) will be deduced from it. Thus from the terms 16, 20, 25, we deduce 16, 36, 81; and one can continue in this way to infinity, so that from the mean of these proportions one can form others by the same method. We do not need to develop this subject further.

LII. In the same way that all these proportions and all their ratios are composed of the first ratio of equality, so also are they definitively resolved in it. In fact, if any proportion of three une-

qual terms is given, we subtract the smallest term from the middle term, and from the largest, the smallest and twice the middle term diminished by the smallest. If, next, we put the terms thus obtained in order, we will have the same smaller term for the first term, then for the second, the excess of the mean term over the smallest term and finally for the third term, that which remains from the largest. The proportion which will result from this breakdown will itself be the same that give birth to the new proportion. When this breakdown is repeated, one will arrive at the proportion of equality which is the first origin of all proportions and which cannot itself be resolved into any other, but only into the ratio of equality.

Eratosthenes demonstrates that all figures result from some proportion, and that in order to construct them it is necessary to start from equality, and that they resolve themselves in equality. It is not necessary for us to go further on this subject.

SHAPES

LIII. We will find the same results in the shapes, the first of which is the point, which is a spot without extension, without dimension, being the limit of a line and holding the same place as unity does in numbers. The magnitude which has only one dimension and is divisible in only one manner is the line, which is a length without width. The magnitude extended in two directions is a surface, having length and breadth. The magnitude having three dimensions is the solid, which has length, width and height. Now the solid is contained and defined by surfaces, the surface is defined by lines and the line is defined by points.

Among lines, the straight line is that which is direct and as though stretched; it is that which, between two given points, is the shortest of all lines having the same extremities, and which is extended equally between all its points. The curved line is that which does not have this property. The same difference is found between the plane and the curved surface. In fact, surface is the apparent limit of all solid bodies, following two dimensions, length and breadth. Now the plane is in a straight surface such that if a straight line touches it at two points, it coincides with it throughout its whole length. Straight lines are parallel when, prolonged to infinity on the same plane, they do not meet and always maintain the same distance between them.

Plane shapes are those in which all the lines are in the same plane. Rectilinear shapes are those which surround straight lines, and non-rectilinear shapes do not have this property. Among the plane and rectilinear shapes, those which are contained between three sides are called trilateral. Those of four sides are called quadrilateral, and those which are contained by a greater number of straight lines are called polygons.

Among the quadrilaterals, those which have opposite sides parallel are called parallelograms, and the parallelograms which have right angles are called rectangles. Angles are right angles when a straight line, meeting another straight line, forms two adjacent equal angles with it. Each rectangular parallelogram is, properly speaking, comprised between sides which form right angles, and among these rectangles those which have four equal sides are properly called squares. Those which are not in this category are called heteromecic (unequilateral).[50]

LIV. Among the solids, some to the number of six, are included under planar parallelograms, and are called parallelopipeds. Others are included under rectangles and are called rectangular parallelopipeds. Of those, some are equilateral in all directions, that is to say that the length, the width and the height are equal and these are included under the equal squares and are called cubes. Those which have length and width equal, that is to say square bases, but whose height is less, are called plinths. Those whose length is equal to the width, but whose height is larger, are called beams. Finally those which have three unequal dimensions are called scalene parallelopipeds.

PROPERTIES OF THE MEANS

We now have to speak in more detail of the means, the theory of which is indispensible for understanding the writings of Plato. There is a mean when, between two homogeneous unequal terms, one takes another homogenious term such that the excess of the first, which is at the same time the larger, over this mean term, is to the excess of the mean term over the smaller, as the first term is to itself or to one of the two others, or as the smallest is to one of the other two.

[50] See the definition of Heteromecic (unequilateral) numbers, I, XVII.

LV. In particular, the arithmetic mean is the one in which the mean term is greater than one extreme and is less than the other by the same number, as in the proportion 3, 2, 1. In fact, the number 2 is greater than 1 by one unit and is less than 3 by one unit. This mean term has the property of being the half-sum of the extremes; so that 3 + 1 = 4, which is the double of the mean term 2.

LVI. The geometric mean, also called the proportion proper, is the one in which the mean term is greater than one extreme and is less than the other by a multiple or superpartial ratio (of the first term to the second or of the second to the third), as in the proportion 1, 2, 4. Four is indeed the double of 2, and 2 is the double of the unit, and likewise, the difference 2 - 1 is 1, and the difference 4 - 2 is 2. These numbers, compared with one another, are thus in double ratio. This mean possesses the property that the product of the extremes is equal to the square of the mean term: thus in the preceding proportion, the product of the extremes is 4, since 1 x 4 = 4, and the square of 2 is also 4, for 2 x 2 = 4. Therefore the product of the extremes is equal to the square of the mean term.[51]

LVII. An harmonic proportion occurs when, being given three terms, the first is to the third in the same relationship that the excess of the first (over the second) is to the excess of the second (over the third). Such are the numbers 6, 3, 2: the extreme 6 is the triple of 2, and the excess of 6 over 3 is 3, which is the triple of the unit, which is the excess of 3 over 2. This proportion possesses the property that the mean term is greater than one extreme and is less than the other by the same part of the extremes. Thus, in the proportion formed of the numbers 2, 3, and 6, the extreme 6 is greater than 3 by half of 6, and the other extreme 2 is less than 3 by half of 2. Furthermore, if the extreme terms are added and the sum is multiplied by the mean term, a number is found which is double the product of the extremes. Thus 6 + 2 = 8, and 8 multiplied by the mean term 3 gives 24; now 6 x 2 = 12 whose double is 24.[52]

LVIII. The mean called "subcontrary to the harmonic" (mean) is the one in which the third term is to the first as the excess of

[51] Theon habitually verifies the stated proposition in a simple manner. If we have a, b, c, the three numbers which give the geometric median, by hypothesis then, $a - b : b - c = b : c$, from which it follows that $ac - bc = b^2 - bc$, and consequently $ac = b^2$.

[52] The harmonic median is in general $a - b : b - c = a : c$. In the product of the extremes being equal to the product of the means, we have $(a + c)\,b = 2\,ac$, which demonstrates the stated proposition.

the first (over the second) is to the excess of the second (over the third). Such is the mean formed by the numbers 6, 5, 3, in which 6 is greater than 5 by one unit, and 5 is greater than 3 by 2 units, and in which, finally, 3 is half of 6, as the unit, the excess of the first number (over the second) is half of 2, the excess of the second number over the third.

LIX. We have the fifth mean when, being given three terms, the third is to the second as the excess of the first (over the second) is to the excess of the second (over the third). Such is the proportion formed of the numbers 5, 4, 2. The extreme 5 is greater than 4 by one unit, and 4 is greater than the other extreme 2 by 2 units. Now the extreme 2 is half of 4, and the unit, the excess of the first term (over the second), is half of 2, the excess of the second (over the third).

LX. We have the sixth mean when, being given three terms, the second is to the first as the excess of the first (over the second) is to the excess of the second (over the third). Such is the proportion formed of the numbers 6, 4, 1. The extreme 6 is indeed greater than 4 by 2, and 4 is greater than the other extreme 1, by 3, and 4 is to 6 as 1 is to $1 + \frac{1}{2}$. Now 2, which is the excess of 6 over 4, is to 3, the excess of 4 over 1, in the same relationship as 1 to $1 + \frac{1}{2}$.

The Pythagoreans have been much concerned with these six means and their subcontraries. For us, it is sufficient to have, according to the method of Pythagoras, a condensed outline of these principles, in order to summarize the exposition of mathematics.

HOW THE MEAN TERMS OF THE MEANS ARE FOUND

LXI. This is how one finds the mean terms. In the arithmetic proportion, one adds to the small term, half of the excess of the larger over the smaller; or one adds the halves of each of the two numbers given; or finally one takes half of the sum of the two given terms. If we propose to find the mean term, in arithmetic proportion, between the numbers 12 and 6, we take the excess of the larger 12, over the smaller 6, the half of which is 3, which is added to the smaller term 6, and obtain 9, which is the arithmetic mean between the numbers 12 and 6, since it is greater than one and less than the other by 3 units. Likewise if we add the extremes 12 and 6, the sum is 18, whose half, 9, is the mean between the given numbers.

Here is how the mean term of a geometric proportion is found: we take the square root of the product of the extremes. If, for example, the two numbers 24 and 6 are given, between which we are to find the mean term in geometric proportion, we multiply the given numbers by each other. The product is 144, of which the square root, 12, is the mean term, because 24 : 12 = 12 : 6, in double ratio. If the number contained in the extremes is squared, each mean term found is rational and its length is commensurable with the extremes, and composed of whole units. But if the number contained in the extremes is not a perfect square, the mean term will only be commensurable with the extremes in power.

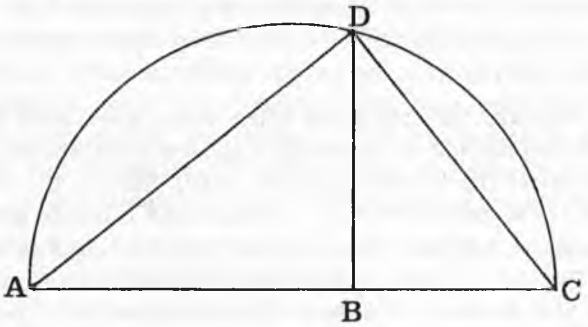


Fig 1

Most often it is determined geometrically whether it can be expressed in rational number or whether the ratio and the lengths are incommensurable. It is done in the following way, taking AB and BC for the two terms. We place them in a straight line and on the sum AC describe a semi-circle, then from point B we lead the perpendicular BD to AC, until it meets the semi-circle. I assert that BD will be the proportional geometric mean between the straight lines AB and BC. In fact, if we join AD and DC, we have a right angle at D, since it is inscribed in a semi-circle. In the triangle ADC, the height is DB and the triangles which are part of it are similar to the total triangle, and consequently similar to each other. Thus the sides which contain the equal angles are proportional, and we have AB : BD = BD : BC, therefore BD is the proportional mean between AB and BC. This is what it was necessary to demonstrate.

It remains now to show how the mean term in harmonic proportion is obtained. We are given two extremes in double ratio, such as 12 and 6. We multiply the excess of the larger over the smaller,

that is to say 6, by the smaller, 6, then divide the product, 36, by the sum of the extremes, that is to say 18, and add the quotient 2 to the smaller term 6, obtaining 8 which will be the sought-for mean, since it is greater than one extreme and less than the other by the same fraction of the extremes, which is a third. The harmonic proportion is then formed of the numbers 12, 8, 6.

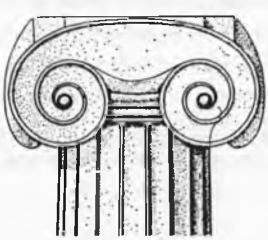
If the given extremes are in triple ratio, such as 18 and 6, we multiply the excess of the larger over the smaller by itself: 12 x 12 is 144, half of which equals 72. We divide this result by the sum of the extremes, or 24. The quotient of the division, added to the smaller term, gives 9 for the sought-for mean term, for it is greater than one extreme and is less than the other by half of the extremes. We have the harmonic proportion of the numbers 18, 9, 6.

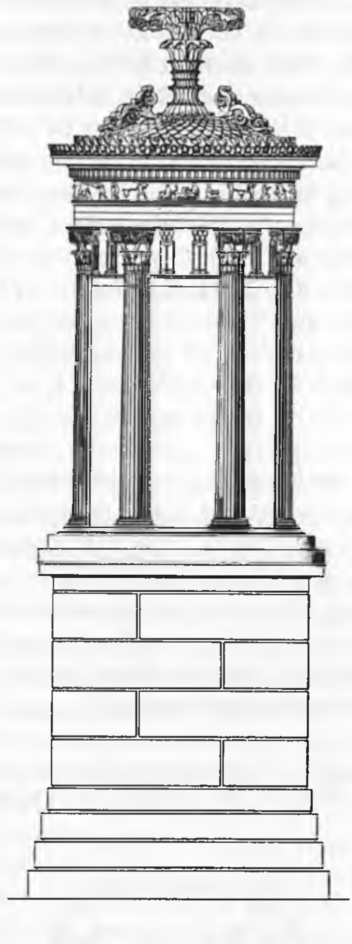
In order to find the harmonic mean between any two given unequal terms, one can also use the more general method that we first demonstrated. By this method it is necessary to multiply the excess by the smaller term and to divide the product by the sum of the extremes, then add the quotient to the smaller term. If we have, for example, the two terms 12 and 4, by multiplying the excess of 12 over 4, that is to say 8, by the smaller term 4, we have 32 as a product. If now we divide 32 by the sum of the extremes which is 16, we obtain a quotient of 2. This quotient 2, added to the smaller term 4, gives 6 for the harmonic mean between 12 and 4. In fact 6 is greater than one extreme and is less than the other by the same fraction of the extremes, which is the half. We therefore have the harmonic proportion of the numbers 12, 6, 4.[53]

After this summary exposition, for the convenience of the readers of Plato, of what is most necessary and useful in the portions of the mathematical sciences of which we have spoken, it remains for us to mention the elements of astronomy.[54]

[53] See note XVI.

[54] After the word 'astronomy' the transcriber of the Venice ms. has added the words "Glory to God"- The transcriber of another ms. adds the words 'end of the present book by the help of God'. (Toulis)





PART THREE

ASTRONOMY

ON THE SPHERICAL FORM OF THE EARTH

1. The entire world is a sphere and the earth, which is itself a spheroid, is placed in the middle.[1] That the earth is the center of the universe and that it is but a point in relationship to the size of the universe: this is what must be established before anything else. A precise exposition of this doctrine would require such lengthy consideration of so many writings that it would instead be sufficient to summarize here what we have to say, by recalling the summary notions transmitted to us by Adrastus.

We would say therefore that the world and the earth are spherical and the earth is the center of the world, and that it is only a point in it. This results from the fact that, for inhabitants of the same place, all the celestial bodies rise and set and rise again at the same points, and always accomplish the same revolutions.

The sphericity of the world is again demonstrated by reason of the fact that, from each part of the earth, as far as our senses can tell, half of the sky is seen above us, while we assume the other half to be hidden by the earth and not able to be perceived. Further-

[1]It can be speculated that the concept of Earth as center of the universe is retained by the Pythagoreans for its philosophical and spiritual verity, and is in a certain sense verified by the theory of relativity as applied to the movement of celestial bodies. But certainly from a spiritual point of view a geocentric universe is a complementary concept, not a contradiction to the heliocentric view.

more, if we look at the extreme points of the sky, all the visual rays appear equal to us, and if diametrically opposed stars describe a great circle, one is setting while the other is rising. If the universe, instead of being spherical, were a cone or a cylinder, or a pyramid or any other solid, it would not produce this effect on earth: one of its parts would appear larger, another smaller, and the distances from earth to heaven would appear unequal.

II. And first of all, the earth is spherical from east to west, the rising and setting of the same stars certainly prove this. They take place early for the inhabitants of eastern regions, and later for those of the western regions. A single lunar eclipse further shows this: it is produced in the same brief period of time, but for all those seeing it, it appears at different hours of the day. The further eastward one is, the more advanced the hour will be that one sees it, and the sooner one will have seen a greater part. Because of the curved form of the earth, the sun does not illuminate the whole surface of the earth at the same time, and the shadow that the earth projects moves according to a fixed order, the phenomenon taking place at night.

It is again evident that the earth is convex from north to south: in fact for those going southward, for the measure that they advance, many stars, which are always visible for us in their movement around the pole, have a rising and a setting. In the same way, other stars, always invisible for us in their movement around the pole which is hidden for us, have for them a rising and a setting. Thus the star called Canopus [2] is invisible in lands further north than Cnide [3]; but it is visible in more southerly lands, and always rises higher and higher to the measure that one is further from the north. On the contrary, when one goes from south towards the north, many of the stars which are seen rising and setting in the south disappear, entirely, while others, situated in the region of the Bears, and which used to have a rising and a setting, become always visible, and as many more of these are seen as one advances northwards.

Since the earth appears convex from all parts, it must be spherical. Furthermore, every weighted body is carried naturally toward the center. If we concede that certain parts of the earth are further

[2]from the constellation of Argo, one of the most brilliant stars in the southern hemisphere. According to ancient lore, Canopus was the captain of Menelaus' boat (brother of Agamemnon and husband of Helen) when they sailed against Troy.

[3] City of Carie (Asia Minor.) (Toulis)

away from the center than others because of their size, it would be necessary that the small parts which encircle them be pressed, repelled and removed from the center, to the point where equality of distance and pressure being obtained, everything being constituted in equilibrium and repose like two wooden beams which mutually support each other or two athletes of the same force, mutually held in a clasp. If the different parts of the earth are equally far from the center, it is necessary that its form be spherical.

In addition, since the fall of heavy bodies is always and everywhere made towards the center, all converge towards the same point and each falls vertically, that is to say that angles which are always equal are made with the surface of the earth, then it must be concluded that the surface of the earth is spherical.

III. The surface of the sea and all tranquil waters are also spherical. This can be recognized in this manner: If, situated on the shore, one observes an object from which one is separated by the sea, such as a hill, a tree, a tower, a ship, or the earth itself, then if one lowers one's gaze towards the surface of the water, one no longer sees anything, or one sees a smaller part of the object, the convexity of the surface of the sea masking the object. And often, while sailing, from the bridge of the ship neither earth nor any other vessel can be seen, but a sailor climbs up high on a mast pole and can see them, being higher and so overcoming the convexity of the sea which was causing the obstacle.

It can be physically and mathematically demonstrated that the surface of every body of still water must be spherical in form. Water indeed tends to flow from higher towards lower levels. But the higher parts are further away from the center of the earth, and the lower parts are less so. The surface of the water being presumed planar, let us call ABC a straight line on this surface. From the center of the earth, such as point X, we draw the perpendicular to the base XB, and we draw the lines XA and XC to the extremities of this base. It is evident that the two lines XA and XC are both longer than XB, and that the two points A and C are further from the center than point B, and consequently, higher than point B. Water will flow therefore from points A and C towards the lower point B, until the latter point, surrounded with new water would be as far from point X as A and B are. Similarly, all points of the surface of the water will be at the same distance from X; therefore

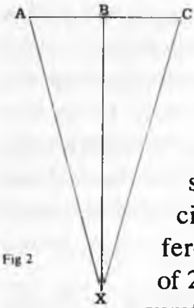


Fig 2

water presents the spherical form and the entire mass of the earth's water is spherical.

It cannot be said that the height of mountains or the depth of valleys would be contrary to this thesis and prove that the earth is not an exact sphere. Erathosthenes indeed shows us that the circle of the earth, measured following the circumference of a great circle, has an approximate length of 252,000 stades, and Archimedes tells us that a circumference of a circle, developed as a straight line, has the value of three times and nearly a seventh of the circle's diameter. The diameter of the earth will then have the value of approximately 80,182 stades. Three times this number, plus a seventh of this number gives, in fact, 252,000 stades.

Now, according to Erathosthenes and Dicearchus, the vertical height of the highest mountains above the lowest plane is 10 stades. They have deduced this result from observations made with the *diopter* [4] which allows the measurement of heights according to certain intervals. The height of the largest mountain would therefore be equal to an eight-thousandth part of the total diameter of the earth. If we were to make a sphere one foot in diameter, the width of one finger being about equal to $12^1/_2$ diameters of a grain of millet, the diameter of our sphere would equal 200 millet-grain diameters or a little less. Since the foot has the value of 16 fingers [5], the finger has the value of 12 millet-grain diameters, and 16 times 12 is 192. One quarter of the diameter of a grain of millet is then larger than the eight-thousandth part of a foot, since 40 times 200 is 8,000.

But we have seen that the height of the largest mountain is nearly an eight-thousandth part of the diameter of the earth, thus the relationship of a quarter of the diameter of a grain of millet to this one-foot sphere is greater than the relationship of the height of the highest mountain to the diameter of the earth. And the relationship of the sphere having a quarter of the thickness of a grain of millet to the sphere of a foot in diameter is greater than the relationship of the sphere of 10 stades in height to the height of the terrestial sphere.

The sphere having a diameter of a quarter of the diameter of a grain of millet is the 64,000th part of a whole grain. The spherical

[4]A kind of graphometer.

[5]Finger — the smallest of the ancient Greek measures of length equal to 0.018 of the meter. (or .0196 of a yard) Toulis.

mountain of 10 stades in diameter has the value of nearly 524 cubic stades, and the whole earth as a sphere, has the value, in cubic stades, 270 third myriads, 250 second myriads, 4350 first myriads, 8296 and the fraction $^{11}/_{21}$. [6]

Furthermore, it is demonstrated that the rectangle formed by the diameter of a sphere and the circumference of a great circle, opened out as a straight line, equals 4 times the surface of a quarter of the sphere, and this quarter equals the area of the circle. The square of the diameter is to the area of the circle as 14 is to 11 [7]; for the circumference of the circle equals three times the diameter plus a seventh part of this diameter. If the diameter is seven, the circumference is 22. A quarter of this circumference is 5 + $^1/_2$. Therefore, the square of the diameter being 49, the circle having this diameter is 38 + $^1/_2$; and if we double this in order to make the $^1/_2$ disappear, the square of the diameter being 98, the circle having this diameter will be 77. However the relationship of these numbers, expressed in the smallest prime terms, is that of 14 to 11, since the largest common measure of these numbers is 7, which is contained one and a half times in the cylinder, according to Archimedes, is also equal to the relationship of 14 to 11. Thus,then, when the cube of the diameter of the circle equals 14, the circumscribed cylinder will equal 11 and the sphere 7 + $^1/_3$.

It is thus that one finds volumes expressed in numbers of the terrestrial sphere and of the highest mountain. If a mountain 10 stades high were a sphere, it would be much smaller, with respect to the earth, than $^1/_{64,000}$ th of a grain of millet in relation to a sphere of one foot in diameter. But mountains are not spherical, and, as we see them, they are much smaller. But one such part of a grain of millet, whether it be superimposed on a sphere of one foot in diameter, or whether it be picked up and placed in a hollow, will not produce any difference in form. The highest mountains of 10 stades have the same relationship with the earth, they will not therefore prevent the whole of earth and sea from truly being a sphere.

[6]First myriads are 10,000 units; seconds are 10,000 times 10,000 or 10,000,000 units; thirds are 10,000 times 100,000,000 units. The above number is written, in our form of enumeration: 270,025,043,508,297 and $^{11}/_{21}$.

[7]The ratio of 14 : 11 for finding the area of the circle is attributed to Archimedes, but can be derived as well from the approximately 7 : 11 ratio of the vertical half-section of the Gizeh pyramid. Both ratios give an approximation of the $\sqrt{\phi}$

The encircling of the earth therefore is valued. 252,000 stades,
the diameter. . . . . 80,182 stades,
the square of the diameter. . . . . 6,429,153,124 square stades,
the cube. . . . . 515,502,355,788,568 cubic stades,
a fourteenth of this cube. . . . . . 36,821,596,842,040 and $^4/_7$ ths.

The product of this number by $^{22}/_3$ is equal to the volume of the earth which therefore has the value in cubic stades of 270,025,-043,508,297 and $^{11}/_{21}$ [8]

IV. The earth is spherical and placed at the center of the world. If it were removed from this position it would not have half the sky above it and half below. Furthermore, straight lines drawn from any point at the extremities of the celestial sphere would not be equal. That the volume of the earth has no perceptible relationship with the expanse of the universe, that it occupies but a point in this universe, is shown by the points of the sundials in every inhabited place on earth. They can in fact be taken for the center of the solar orbit, for in changing location, one cannot observe any perceptible parallax. If therefore there is necessarily a center for the ensemble of all spheres, all the points on earth appear to be this center. It is then evident that the whole earth is only a point with respect to the entire sphere of the sun and even more so with respect to the sphere of the stars. It is for this reason that half of the world, or nearly so, is always apparent to our eyes.

Although we might say many other things about the form of the universe and of the earth, and of earth's central position as well as its size, and the unknown size of its relation to the universe, what has been demonstrated by Adrastus in the manner shown above will be sufficient for the exposition of what follows. Here is what he next says:

ON THE CELESTIAL CIRCLES

V. The celestial sphere turning around its immobile poles and the axis which joins them, at the middle of which is fixed the earth, and all the stars carried by this sphere and all the points of the sky, describe parallel circles, that is to say circles everywhere equidistant, perpendicular to the axis, and drawn from the poles of the universe as centers. One can count the circles described by the stars, but the circles described by the other points are innumerable. Some of

[8]For the rectification we have made of the values of the different results, See note XVII.

these circles have been given particular names which it is useful to know in order to take into account what happens in the heavens.

There is one above us, around the always apparent and visible pole. It is called the arctic circle, because of the constellations of the Bears which it crosses. Another, on the opposite side, equal to the first, circles the pole which we never see, is itself always invisible to us, and is called the antarctic circle. The one in the middle, which is a large circle, divides the whole sphere into two equal parts and is called the equator, because on these corresponding regions of the earth, there is an equality between the days and nights; for other places in which one sees the sun rise and set according to the general movement of the universe, the durations of the day and night are equal when the sun describes this circle.

Between the equatorial circle and the two arctic circles there are the two tropics, the tropic of summer, situated for us on this side of the equatorial circle, and on the other side, the tropic of winter. The sun, in its revolution, sometimes moves near to one and sometimes to the other. The zodiac is in fact obliquely extended between these two circles.

VI. The zodiac is also a great circle. It touches each tropic at one point: the tropic of summer at a point in Cancer and the other tropic at a point in Capricorn. It cuts the equatorial circle into two equal parts and is itself also divided by this circle at a point in Aries and at a point in Taurus. It is in this zone that the sun, moon, and other planets move: Phenon, which the planet of Saturn is called by some, like the Sun; Phaethon, the planet of Jupiter; Pyrois, the planet of Mars or as others claim of Hercules; Phosphorus,[9] which is also called Venus or also Lucifer[10] and Hesperus;[11] and close to these planets Stilbon, "the shiny one," which is also called Hermes. (Mercury)

VII. The circle which is the boundary of our vision and divides the sky as a whole into two equal parts — the earth being the obstacle from our viewpoint — is called the horizon. Of these parts the one above the earth is the visible hemisphere, and the other below is the invisible hemisphere. As it is also a great circle of the sphere, it also cuts the great circles such as the equator and the zodiac into two equal parts. If two stars are diametrically opposed, when one

[9]Literally meaning the light-bringer, one who brings light. (Toulis)

[10]1. Morning — bringer of dawn

[11]2. Evening — Bringer of evening.

rises, the other sets. The horizon divides the meridian into two equal parts.

VIII. For there is another great circle, called the meridian, which passes through the two poles of the world, and which is conceived as perpendicular to the horizon. It is called meridian because the sun cuts it at the middle of the day, being at the highest point of its course above the horizon. It is sometimes called "truncated" [12], because one of its parts, that which is on the side of the invisible pole, is hidden from us.

IX. The equatorial (circle) and the two tropics situated on either side of it are circles which are given and fixed in size and position. It is said that points and lines are 'given' in position when they always occupy the same place; it is said that surfaces, lines and angles are given in size when equal sizes can be found. Now the equatorial circle and the two tropics placed on either side of it always have the same position, they are always fixed, and one could find equal circles: the zodiac, the horizon and the meridian being equal to the equatorial circle, and the tropic of summer being equal to the tropic of winter and reciprocally. This is why they are always given; it is not in our power to render them as this or that; they are naturally so, they are given, we cannot determine them.

But those which it is in our power to render as this or that are not naturally given. Those which are naturally given, that is, which are fixed and exist by themselves, are the equatorial circle and the circles situated on either side of it, given in size and position. The zodiac is a circle given in size and position with respect to the sky, but in relation to us, it is not given in position. Indeed, for us it is not fixed, because of its obliquity in the universe, which to us shows it changing in place.

The meridian and the horizon are also given in size, for they are the great circles of the celestial sphere, but they change in position according to the earth zones and are different in different locations on the earth. In fact we do not all have the same horizon nor the same zenith, nor the same meridian. As for the arctic and the antarctic circles which are neighbors of the poles, they are not given in either size or position,[13] but according to the differences in more northerly or southerly zones, they are seen to be larger or

[12]Colure from Κολουρος truncated.

[13]One would call the arctic circle in each location the parallel limit of the stars always visible in that place, and the antarctic circle the parallel limit of the always invisible stars.

smaller. But for the middle region of the earth, that is for the zone which is found on the equatorial line where one cannot live because of the heat, it is not the same: the two poles appear at the end of the horizon, and it is sometimes said that the sphere is right because in this region of the earth all the parallel circles are perpendicular to the horizon.

X. Each of the other circles is a true circle terminated by a single line, but the one called the zodiac shows a certain breadth, like the cylinder of a drum. The animal figures are imagined on the inside of this cylinder. The circle in the middle of these signs is called the great circle which touches the two tropics at one point on each of them, and cuts the equatorial circle into two equal parts. The two circles which define the width of the zodiac on either side are smaller circles.

THE STARS

XI. Most of the stars are fixed; they are carried together by a unique and simple circular movement, with the first sphere which is the largest, as if they were fixed to it and as if they were moved by it. They always have the same relative position on the sphere, and maintain the same order between each other and do not experience any change of form or movement, nor of size or color.

THE PLANETS

XII. The sun, moon and other stars which are called errant are carried with the universe in the diurnal movement from east to west, just like the fixed stars. But apart from this movement, each day they appear to have several other complicated motions. For, through a movement of their own, they go toward the zodiac signs which follow them (in the diurnal movement) and not to the zodiac signs which precede them, carried along in the contrary direction of the universe in a course which is called movement in longititude. In addition, they have a movement in latitude, from north to south and reciprocally, while accomplishing their course in the contrary direction to the movement of the universe. Attentive observers see them moved from the tropic of summer to the tropic of winter and reciprocally, through the obliquity of the zodiac.

And within the breadth of the zodiac, they are sometimes seen further north from the middle circle, and sometimes further south;

some are lower down, others less so. Besides, they vary in size, sometimes being further distant, and sometimes being closer to the earth in the depths of space. It is for this reason that the speed of their movement through the zodiacal signs appears unequal. They do not cover the same distance in space in the same amount of time; they go faster when they appear larger because of their lesser distance from earth, and they go less fast when they appear smaller because of their greater distance.

The distance covered on the zodiac is slight for the sun, since it is just about one degree out of 360. For the moon, as the ancient astronomers have said, and for Venus, it is larger, since it is about 12 degrees. Hermes covers about 8, Mars and Jupiter about 5, and Saturn nearly 3. The moon and the sun each appear to deviate equally in latitude from the circle of the middle of the zodiac. The other planets do not deviate equally from it, but are more northward in one sign, and more southward in another.

As for the length of the circle of the zodiac, from one fixed point to this same point, the moon, going towards the same zodiacal signs and not towards the preceding ones, travels it in $27^1/_3$ days, the sun in a year having the approximate value of $365^1/_4$ days. Venus and Hermes travel in an unequal movement, but with little difference in duration, and it can be said that they have the same speed as the sun, since they are always seen beside it, sometimes following it, sometimes preceding it. Mars achieves its course in a little less than 2 years, Jupiter in about 12 years, and Saturn in a little less than 30 years.

The conjunctions with the sun, the appearances and disappearances which are called the risings and the settings, are not the same for all the planets. The moon, in fact, after its conjunction with the sun, having a more rapid movement than the sun towards the zodiacal signs which follow, appears first and rises in the evening, while it disappears and sets in the morning. Inversely, Saturn, Jupiter and Mars, which arrive less quickly than the sun at the following signs, are preceded and overtaken by it, that is to say, that these planets always set in the evening and rise in the morning (after the conjunction).

XIII. Venus and Hermes, which have a movement equal to that of the sun, always appear near to it; sometimes these two stars follow it, sometimes they precede it; sometimes they appear in the evening and disappear also in the evening, sometimes they appear

at the earliest dawn and disappear with the day. While the other planets are far from the sun, at every interval, up to the point where they are diametrically opposite, these two stars are on the contrary always seen near the sun. Hermes is about 20 degrees away from it, that is, about two thirds of a zodiacal sign, either to the east or to the west. Venus is about 50 degrees to the east or to the west.

XIV. Rising means several things. First, properly and commonly, for the sun and the other stars, by their elevation above the horizon; next, for their shining to begin to distinguish itself from the rays of the sun, which is still properly a manner of rising. There still remains the rising called rising at "night-edge" [14], which is produced in the east after the setting of the sun, in the part of the sky diametrically opposite. It is called "night-edge" because it occurs at the edge of the night, i. e. at its beginning. Similarly the first setting is the descent below the horizon. Next is the setting produced by the diffusion of the brilliance of the star by the luminous rays of the sun; which is properly called a disappearance. There still remains the setting called again night-edge, but at dawn, when the sun rises, a star disappears in the part of the horizon diametrically opposite.

Among the risings and the settings depending on the sun and its rays, that is to say among the phenomena of appearance and disappearance, some occur in the morning, others in the evening. The rising of the star belongs to the morning when the star preceeding the rays of the sun appears before it in the east, as with the rising of Canis Major. The rising belongs to the evening when the star begins to appear after the setting of the sun, as we have said of the new moon. Similarly, the setting belongs to the morning when the star, which in the preceding days was rising before the sun, like the moon, ceases to appear at its approach; the setting is of the evening when the sun, being very close to a star in the west, causes that star to become invisible because of the radiance of its neighbor.

THE ORDER OF THE PLANETS AND THE CELESTIAL CONCERT

XV. Here are the opinions of certain Pythagoreans relative to the position and the order of the spheres or circles on which the planets are moving. The circle of the moon is closest to the earth, that of

[14] "Night-edge" here applies to the *early beginning* and *early end* of the night, i. e. at early dawn and early sunset. (Toulis)

Hermes is second above, then comes that of Venus, that of the sun is fourth, next comes those of Mars and Jupiter, and that of Saturn is last and closest to that of the distant stars. They determine, in fact, that the orbit of the sun occupies the middle place between the planets as being the heart of the universe and most able to command. Here is a declaration of Alexander of Aetolia:

"The spheres rise higher and higher;
the divine moon is the nearest to the earth;
the second is Stilbon, 'the shining one', star of Hermes, the inventor of the lyre;
next comes Phosphorus, brilliant star of the goddess of Cythera (Venus);
above is the sun whose chariot is drawn by horses, occupying the fourth rung,
Pyrois, star of the deadly Mars of Thrace, is fifth;
Phaeton, shining star of Jupiter is sixth;
and Phenon, star of Saturn, near the distant stars, is seventh.
The seven spheres give the seven sounds of the lyre
and produce a harmony (that is to say, an octave), because of the intervals which separate them from one another."

According to the doctrine of Pythagoras, the world being indeed harmoniously ordained, the celestial bodies which are distant from one another according to the proportions of consonant sounds, create, by the movement and speed of their revolutions, the corresponding harmonic sounds. It is for this reason that Alexander thus expresses himself in the following verse:

"The earth at the center gives the low sound of the hypate;
the starry sphere gives the conjunct nete;
the sun placed in the middle of the errant stars gives the mese;
the crystal sphere gives the fourth in relation to it;
Saturn is lowest by a half-tone;
Jupiter diverges as much from Saturn as from the terrible Mars;
the sun, joy of mortals, is one tone below;
Venus differs from the dazzling sun by a trihemitone;
Hermes continues with a half-tone lower than Venus;
then comes the moon which gives to nature such varying hue;
and finally, the earth at the center gives the fifth with respect to the sun; and this position
has five regions, from wintry to torrid,
accomodating itself to the most intense heat, as to the most glacial cold.
The heavens, which contain six tones, complete the octave.
The son of Jupiter, Hermes, represents a Siren to us,

having a seven-stringed lyre, the image of this divine world."[15]

In these verses Alexander has indicated the order for the spheres that he has determined. It is evident that he arbitrarily imagined the intervals which separate them, and nearly all the rest. Indeed, he says that the seven-stringed lyre, the image of the universe, was constructed by Hermes, and that it gives the consonances of the octave; then he established the harmony of the world with nine sounds which, however, include only six tones.

It is true that he attributes the sound of the hypate, as being lower than the others, to the earth; but being immobile at the center, it renders absolutely no sound. Then he gives the sound of the conjunct nete to the sphere of the stars, and places the seven sounds of the planets between the two. He attributes the sound of the mese to the sun. The hypate does not give the sound of the fifth with the mese, but that of the fourth, and it is not with the nete of the conjuncts that it gives the consonance of the octave, but with the nete of disjuncts.

The system does not conform to the diatonic type, since the song of that gender allows for neither a complete trihemitone interval, nor two half-tones one after the other. Neither is it chromatic, for in the chromatic gender, the melody does not include the unbroken tone. If it be said that the system is formed of the two genders, I would answer that it is not melodious to have more than two half-tones following one another. But all of this is unclear to those who are not initiated into music.

Eratosthenes, in a similar manner, exposes the harmony produced by the revolution of the stars, but he does not assign the same order to them. After the moon which is above the earth, he gives the second place to the sun. He says that in fact Mercury, still young, having invented the lyre, first rose up to the sky, and passing near the stars called "errant", he was astonished that the harmonies produced by the speeds of their revolutions was the same as

[15]This, then, according to Alexander, is the order of the spheres and the intervals of the sounds made by these spheres:

Sphere	Interval	
Sphere of the stars, giving the nete		
	half tone	fourth
Sphere of Saturn		
	half tone	
Sphere of Jupiter		
	half tone	
Sphere of Mars		
	tone	
Sphere of the sun giving the mese		
	trihemitone	fourth
Sphere of Venus		
	half tone	
Sphere of Hermes		
	half tone	
Sphere of the moon		
	tone	
Sphere of the Earth giving the hypate		

that of the lyre which he had constructed....In the epic verses, this author appears to leave the earth immobile and determines that there are eight sounds produced by the starry sphere and by the seven spheres of the planets which he makes circle around the earth. It is for this reason that he made an eight-stringed lyre including the consonances of the octave. This explanation fares better than that of Alexander.

The mathematicians establish neither this order nor a like order among the planets. After the moon, they place the sun, and some put Hermes beyond it, then Venus, and others put Venus, then Hermes. They arrange the other planets in the order we have mentioned.

THE MYTH OF PAMPHYLION IN PLATO'S REPUBLIC

XVI. Plato, at the end of the *Republic*, wishing to exhort justice and virtue, recounts a fable in which, speaking of the arrangement of the celestial bodies, he says that an axis traverses the celestial pole like a pillar. He adds that there is another spine-like axis with hollow vertebra nested one next to the other. These vertebra are none other than the spheres carrying the seven planets. The last sphere being that of the stars, envelopes all the others. He shows the order of these spheres with respect to the distance of each of the stars, to their color and to the speed of their movement in the opposite direction to that of the universe. This is what he says[16]:

> "After each of these souls had passed seven days in the meadow, they had to depart from it on the eighth and continue further through a four day journey to a place from which one could see a light extending over the whole surface of the sky and earth, straight like a pillar, quite similar to a rainbow but brighter and more pure. They made still another day's journey to arrive there, and saw, in the middle of this luminous band, the ends of its fastenings attached to heaven. This band is the link of heaven and embraces its whole circumference, like the undergirders of triremes (in order to prevent the structure from falling apart). At the extremities of this link was held the spindle of Necessity, it is this which gives the oscillation to all the revolutions of the spheres. The shaft and the hook of this spindle were of diamond; the spindle was formed of the same substance and of other precious materials.
>
> "This is how it was made: in its form it resembled the spindle-wheels of our world, but according to the description given by

[16]Plato, *Republic* X 616b.

Pamphylian, it should be represented as containing in its hollow another smaller spindle-wheel, which itself received a third, like large vessels fitted one inside the other. There was thus a third, a fourth and yet four more of them. There were, therefore, eight spindle-wheels in all, placed one within the other. Their circular rims could be seen from above, and they presented the continuous curved surface of a single spindle around the shaft passing through the center of the first. The circular rim of this exterior spindle was the widest, then that of the sixth, the fourth, the eighth, the seventh, the fifth, the third and of the second, diminishing in width in that order.

"The rim of the largest spindle (the sphere of the stars) was of different colors, the rim of the seventh (sphere of the sun) was of a very brilliant color, that of the eighth (sphere of the moon) borrowed its color and its brilliancy from the seventh. The color of the circles of the seventh and the fifth (Saturn and Hermes) was nearly the same and they were more yellow than the others; the third (Jupiter) had a very white color; that of the fourth (Mars) was slightly red. Finally the sixth (Venus) occupied second place in brilliancy and whiteness."

The complete exterior spindle made its revolution in the same direction as the universe, and, in the interior, the seven concentric spindles moved slowly in the opposite direction. The movement of the eighth was the most rapid; the movements of the seventh, sixth and fifth were less of an equal speed; the fourth which has a more rapid retrograde movement than the other spindles is third in speed, as it appeared to them; the third had only the fourth speed and the second had only the fifth. The spindle turned on the knees of Necessity. On each of these circles a Siren was seated, who turned with it and emitted one sound, always the same, to be heard. From all these sounds, eight in number, resulted a perfect harmony (that is to say a complete octave)."

This is what Plato says. We explain this passage in the *Commentaries on the Republic.* Also, we have constructed a sphere according to his explanations. Plato says, in fact, that it would be useless work to wish to expose these phenomena without the images which speak to the eyes. He says that Sirens are seated on the circles, thus some designate the planets themselves by the word "Siriazein," to burn.[17] Of the others, after Adrastus, the poets often call all the

[17]The word "Siriazein", which is neither in the dictionaries nor in the Thesaurus of Henri Estienne, appears to be derived from the word "Sirius", meaning burning, shining, from Sirius: a star that burns hotly — a burning star — also applies to the burning sun when the two words are used in conjunction like "The Sirionic Sun" — when the word is used alone, it means either the star Sirius or the flaming burning heat of any star. (Toulis)

stars burning "Siriuses" stars. Thus one reads in Ibycus: "flaming as the 'Siriuses' which shine in the long night."

Others designate only particularly remarkable and brilliant stars. Aratus uses the 'condition' "Sirius" to indicate that one star in the constellation of the Dog burns with a lively flame [18] and a tragic poet said of a planet, "What then is this shining 'Sirionic' star which passes above our heads"? [19] Some authors pretend that the stars cannot be taken for the Sirens, but that, according to the Pythagorean doctrine, the sounds and the accords are produced by their revolutions, from which results a perfect harmony.

THE MOVEMENTS OF THE PLANETS

XVII. As for the planets, says Adrastus, there are some which are always lagging behind. Such are the sun and moon which never go towards the zodiac signs which precede, but which are always seen going towards those which follow; also these planets never have stations nor retrograde motion. There are others which move towards the preceding zodiac signs and towards the following signs, these are all the other planets. It is for this reason that they necessarily appear sometimes to stop, and sometimes to move in retrograde motion.

XVIII. The contrary movement is an appropriate motion, according to Adrastus, of a planet which seems always to go towards the signs which follow at the east. But according to Plato, this is not an appearance, it is in reality the true movement of a star which goes to the east in consecutive zodiac signs, for example, from Cancer into Leo.

XIX. Forward movement is the apparent motion of a planet which seems to go towards the preceding zodiac signs to the West, for example, from Cancer to Gemini.

XX. The station is the apparent state of a planet which seems to stop and remain some time near one of the fixed stars.

XXI. Retrograde motion is the apparent return of a planet from its station in the opposite direction of its first movement. It all appears to be produced in this way, but this is an appearance. The

[18]Aratus, *Phenomena,* v 331.

[19]Euripedes, *Iphigenia at Aulis,* v. 6-7.

cause is that each planet moves below the stars in a circle or in a sphere of its own, and seems to us, because of the superposition, to be carried back, relative to the zodiacal zone which is above. And as Adrastus explains, these are only the different hypotheses on the planets, hypotheses rendered probable through their accord with the phenomena.

XXII. He says that the world, in that it is composed of parts as numerous and as diverse as we have singled out, moves in a circular movement proper to its spherical form, and that this motion has been initiated by a first propulsion; this is why the world has been arranged by the grace of a superior and benevolent cause. The motion of the planets has been diversely arranged for the calculation of time and for their return to the perigee and to the apogee, in such a way that that which happens here below completely follows this motion. It is indeed through these revolutions of the stars that come and go that all things in our world are also changed. The circular motion of the stars is simple and unique, it is regular and uniform. The motion of the planets is, it is true, circular, but appear to be neither simple and unique nor uniform and regular. And in the sublunar world, around us and up to us, all is change and motion, and as the poet says:

"Here below one sees only wrath and killing
and all the other evils."[20]

Indeed there is nothing but generation and decay, growth and decline, alteration in every kind and changing of place. The planets, says Adrastus, are the cause of all these phenomena. One would not say that that which is the most precious, of the divine, the eternal, the non-engendered and incorruptible, follow that which is lesser, mortal and perishable, but these things are certainly thus because of that which is best, most beautiful, most blessed, and that which is here only follows the motion of higher things incidentally.

In order that the motion of the universe which results from an active force and from a divine cause, be circular and always similar to itself, it is necessary that the earth occupy the center around which the motion is produced. And it is necessary that it be below, and also that fire must occupy the opposite place from the etheric essence which moves in a circle. Between the two elements thus separated, must lie the others, water and air, in proportion. This being

[20]Cf. Empedocles, vs. 19 or 21

so, it is still necessary that there be a change in all things here below, because the nature of things is profoundly changing and is subject to contrary forces.

Change is made by the varied motion of the planets. In fact, if the planets were conveyed following parallel circles by the same motion as the fixed stars, the arrangement of all the bodies being universally the same, there would be no change, no vicissitudes here below. But the solstices and the equinoxes, the movements forward and the returns in height and in latitude, particularly the sun and the moon, but also of the other planets, bring forth the different seasons and produce all transformations and all generations and all alterations in our world. The variety presented by the revolutions of the planets, comes from the fact that, fixed to the circles themselves or the spheres themselves whose motion they follow, they are conveyed through the zodiac. Pythagoras, however, was the first to understand that the planets move according to a regulated revolution, simple and equal, but from which results, *by chance,* an apparently varied and irregular motion.

XXIII. Here is what Adrastus says of the position of the circles or spheres, a position which saves the appearances.

It is natural and necessary that, like the fixed stars, each of the other celestial bodies be conveyed uniformly and regularly, by a simple motion which is appropriate to it. I say this will be evident, if, by thought, supposing the world to be immobile, we imagine that the planets move below the immobile (by hypothesis) zodiac. Their movement then will no longer appear varied and unequal, but will appear to be accomplished regularly as we have shown through the construction of Plato's *sphere.*

A double motion is the cause of the apparent varied movement in both directions the starry sphere is conveyed from east to west around the axis which passes through the poles, and in the rapid motion which is appropriate to it. It takes the planets along with it and describes the parallels that the stars follow. On the other hand, the planets, by a slower motion, are conveyed from setting to rising in unequal times, under the zodiac oblique to the three parallel circles, the tropic of winter, the equatorial circle and the tropic of summer. The motion takes place around another axis, perpendicular to the zodiac which diverges from the axis of the stars by the value of the side of the regular pen-

For an explanation of the *spiral* precession of the equinoxes as shown in the zodiacs of the temple of Dendera (now in the Louvre in Paris) see: *Mythological Astronomy of the Ancients Demonstrated,* by Samson Arnold Mackey, Norwich, 1823 (Wizards 1974)

tadecagon.[21] Plato calls the axis of the planets the 'shaft of the spindle' and also the 'spindle'.

XXIV. Adrastus says that the motion is uniform when the spaces travelled in equal times are equal, without ever increasing or diminishing in speed.

XXV. The motion is regular when the moving object has neither a station nor a retrograde movement, but is carried along equally in always the same direction. However, all the planets appear to us to have something irregular, and certain of them even have something unruly in their movement. What then is the cause of such an apparent behavior? The chief cause is that, being on the spheres or the different circles by which they are carried, they appear to move on the zodiac as we have already said.

THE MOVEMENT OF THE SUN

XXVI. By chance, as has been said above, the seven planets, which have nevertheless a simple movement of their own, describe several and varied circles. This will become clear for us if we consider the most brilliant and largest of these planets, the sun. Let us take ABCD as the zodiac, and O as the center of this circle and of the universe, which is at the same time the earth, and AC and BD will be two perpendicular diameters passing through this point. If point A is at the beginning of Aries, B is at the beginning of Cancer, then C is at the beginning of Libra and D is at the beginning of Capricorn.

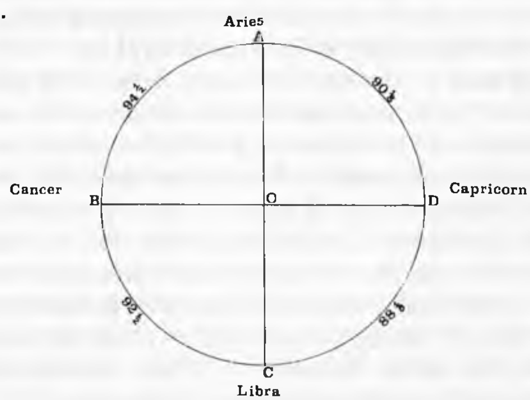


Fig 3

[21] The angle at the center of the regular pentadecagon has a value of a fifteenth of 360 or 24. The angle of the two axes is then 24° according to Theon. This angle is not constant, but its variation is less than a half-second per year. It is now 23° 27'.

The sun is found at A at the spring equinox, at B at the summer solstice, at C at the autumn equinox and at D at the winter solstice. It travels irregularly, in unequal times, the four equal arcs AB, BC, CD, DA. Indeed it moves from the spring equinox to the summer solstice in $94^1/_2$ days, from the summer solstice to the autumn equinox in $92^1/_2$ days, from the autumn equinox to the winter solstice in $88^1/_8$ days, and from the winter solstice to the spring equinox in $90^1/_8$ days, so that it travels yearly the entire circle in approximately $365^1/_4$ days. Its slowest speed is on entering Gemini, and its greatest is in Sagittarius; in Virgo and Pisces it has an average speed.

It is natural and necessary as we have said, that all divine creations (the stars) have a uniform and regular motion. It is then clear that the sun, having a regular and uniform course on its own circle, will appear to us, who look at it from point O of our circle ABCD, to move irregularly. If then this circle were to have the same center as that of the universe, that is to say point O, it would be divided in the same relationships by the diameters AC and BD, we would still be embarrassed by this equality of the angles at the center and by the similitude of the arcs. It is therefore evident that the cause of this appearance is a different movement which does not occur around the center O. The point O will be interior to the circumference, or it will be on the circumference itself, or it will be exterior. Now it is impossible that the solar circumference passes through the point O, for the sun would meet the earth and then some of the earth's inhabitants would have only daytime, others only night time. There would be neither rising nor setting, and one would not see the sun turn around the earth at all, which is absurd.

It remains then to suppose the point O to be at the interior or at the exterior of the solar circle. Whichever hypothesis one chooses, the appearances will be explained, and for this reason one can consider as futile the discussions of mathematicians who say that the planets are conveyed only on eccentric circles, or on epicycles or around the same center as the starry sphere. We will demonstrate that the planets describe *by chance* these three kinds of circles, a circle around the center of the universe, or an eccentric circle or an epicyclic circle. If we suppose that point O is at the interior of the solar circle, but not at the center, it is said that the circle is eccentric; if point O is exterior there is an epicycle.

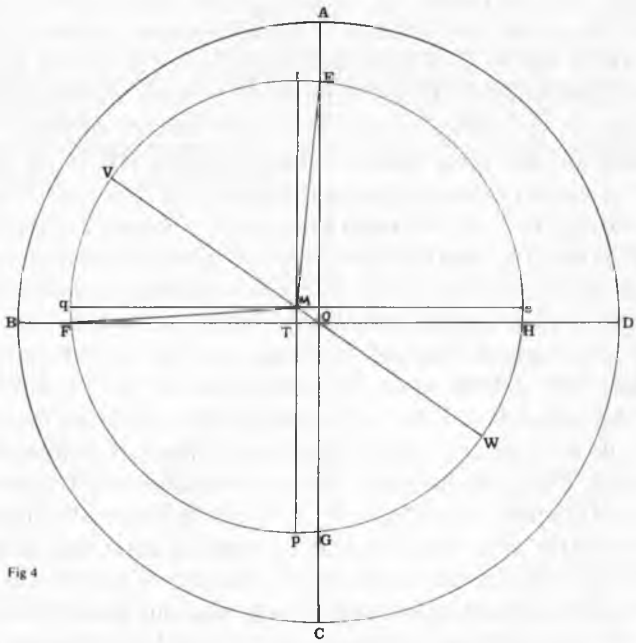


Fig 4

THE ECCENTRIC CIRCLE

XXVI (1). Let us suppose first of all that the eccentric solar circle EFGH is situated in such a way as to have its center under the arc EF, at point M, for example. Let us again suppose that the circle is divided into $365\frac{1}{4}$ equal parts, and that arc EF contains $94\frac{1}{2}$; FG $92\frac{1}{2}$; GH $88\frac{1}{8}$; and HE $90\frac{1}{8}$. It is evident that when the sun is at E, it will appear to us at A, to us that is who see it at a straight line from point O. Then, regularly passing through the arc EF which is the largest of the four divisions of its own circle, in the space of $94\frac{1}{4}$ days, as many days as there are divisions of the arc, it will arrive at F. There it will appear to us at B and it will seem to have irregularly travelled by a number of days different (from a quarter of $365\frac{1}{4}$ days) the arc AB which is a quarter of the zodiac.

Likewise when it travels the arc FG, the second in size of its own circle, in the space of $92\frac{1}{2}$ days which corresponds to the number of divisions in the arc, it will be found at G and will still appear to us in C. It will seem to us that it travelled the arc BC, a quarter of the zodiac, equal to the preceding one, irregularly, in fewer days.

Similarly, when it travels the arc GH, the smallest of the four divisions of the circle, in $88^{1}/_{8}$ days, the number equal to the divisions of the arc, it will be at H and it will appear to be in D to us who observe it from point O. It will seem to us to have travelled the arc CD, equal to the preceding arc, in a fewer number of days.

Finally, for the same reason, when it travels HE in 90 and $^{1}/_{8}$ days, the number of days equal to the number of divisons of the arc, and returning to E, it will seem to us that in 90 and $^{1}/_{8}$ days it travelled the arc DA, equal to the others and that it returned to A. It is for this reason that travelling its circle uniformly, it will seem to travel the zodiacal circle irregularly. But if, joining the centers O and M by a straight line, and extending this line to both sides, we will have MV = MW, since M is the center of the circle EFGH. Thus, the sun at V will be at its greatest distance from the earth, and for us who are at point O, it appears to have its minimum size and speed. This phenomenon appears to be produced at about $5^{1}/_{2}$ degrees of Gemini. Arriving at W, it will be at its smallest distance from the earth and it will appear to have its maximum size and speed. The latter seems to occur at $5^{1}/_{2}$ degrees of Sagittarius. And with reason it appears to have an average size and speed when it occupies the same degrees in Pisces and in Virgo. It is in this way that all the appearances will be explained.

The position and size of circle EFGH is given.Let us now extend through point M the straight lines rp and qs, respectively parallel to the lines AC and BD, perpendicular to each other and joining FM and ME. The circle EFGH is being divided into $365^{1}/_{4}$ parts, it is evident that the arc EFG contains 187 parts and the arc GHE contains $178^{1}/_{4}$ parts, but the arc Er and pG are equal, as are the arcs qF and sH; furthermore, each of the arcs sp, pq, qr is represented by 91 divisions + $^{1}/_{4}$ + $^{1}/_{16}$[22]. The angle rMV is therefore given, and is equal to the angle OMt, that is to say from Mt to Ot is given and the triangle MtO is given in form. But the center O of the universe is also given in relation to the two points V and W, because one of these points is at the greatest distance from the earth and the other at the smallest. The line OM joins the centers of the universe and of the solar circle. The circle EFGH is therefore given in position and in size. It is found by the consideration of the distances and sizes, that the relationship of the line OM to MV is nearly that of 1 to 24. Such is the treatment by the eccentric circle, a treatment which saves (explains) all the phenomena.

[22]Because 91 + $^{1}/_{4}$ + $^{1}/_{16}$ is a quarter of 365 $^{1}/_{4}$.

THE EPICYCLE

XXVI (2). Here is now the explanation according to the epicycle. Again let us take the zodiac ABCD and the solar circle EFGH which is exterior to the center of the universe O. The starry sphere moves from rising B to meridian A, then from point A to D, its setting. The circle EFGH

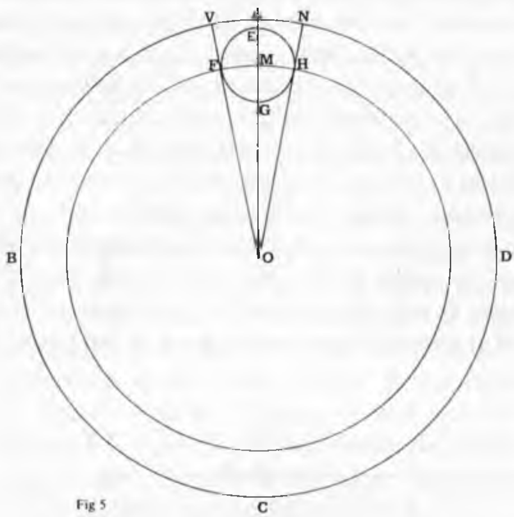


Fig 5

will be either immobile or it will itself move while the sun turns around it. If it were immobile it is clear that the sun would appear neither to rise nor to set, but would always produce day for those who are above the earth and always night for those who are below. In relationship to ourselves and in a single (diurnal) revolution of the universe, it would appear to travel through all the signs. This is contrary to the facts.

The circle then moves, and in moving it is carried either in the same direction as the universe or in the opposite direction. If it turns in the same direction, it is with an equal speed or a greater or lesser one. Supposing that it moves with the same speed, let us draw lines OFV and OHN tangent to the circle EFGH. The sun will always appear to come and go within the arc VAN of the zodiac. In fact, arriving at F, it would appear to be at V and when at E it would appear to be at A, and when carried on to H it will appear to be at N. When it travels through the arc FEH, it will appear to describe the arc VAN towards the signs which precede. Then, when it

travels the arc HGF, it will appear to move through the arc NAV towards the signs which follow. However, this is not what happens. The solar circle EFGH is not carried therefore in the same direction as the universe with the same speed. Furthermore it does not have a greater speed, because then it would appear to overtake the stars and travel through the zodiac in an inverse direction, that is from Aries to Pisces and Aquarius, which is not the case.

It is then evident that the circle EFGH moves in the same direction as the universe with a lesser speed. This is why it appears to be left behind and to pass into the signs which follow, so that it appears to have a movement of its own contrary to that of the universe, everything being conveyed each day in the same direction, from rising to setting. It is thus that it appears to pass into the signs which follow, being in some way left behind.

How then does this save (explain) the phenomena? Taking M as the center of the solar circle, let us describe the circle MrVW from the center O with the radius OM, and let us suppose that the circle EFGH is carried from east to west at the same time as the universe,

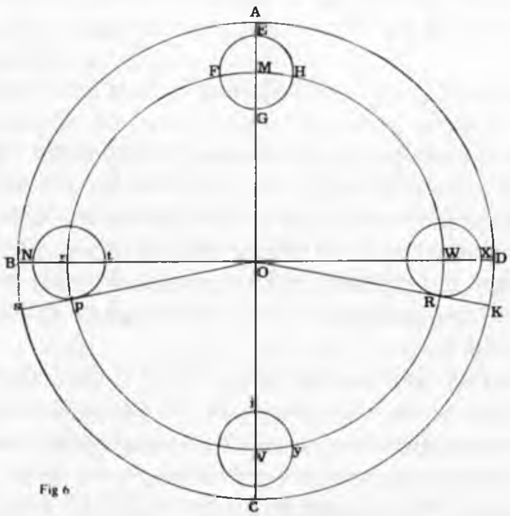


Fig 6

and that it is left behind because of its slower speed, or that it moves in an inverse direction to that of the universe, which appears to Plato as more probable,[23] so that the center, regularly carried on the circle MrVW, transverses it in the space of one year, and the sun also achieves its revolution in the same amount of time, with a

[23]Cf. above, XVIII,

regular movement. In addition, the sun will be carried on the circle EFGH, sometimes in the same direction as the universe and sometimes in the opposite direction, that is to say, in the same direction as its own circle, from point H to point E and from point E to point F. Now, I assert that the circle EFGH, being carried on the circle MrVW in a movement contrary to that of the universe, the sun will move on the circle EFGH in the same direction as the universe and this would explain the phenomena.

Let us first of all suppose that it be carried in a movement contrary to that of the universe, but in the same direction as its own circle, that is to say from E to F, (Fig. 6) and from F to G and from G to H. Then, arriving at E, it will be furthest away from us. It is clear that A will be at 5½ degrees of Gemini[24], therefore C will be at 5½ degrees of Sagittarius. Let us suppose that point M, the center of the solar circle, describes by a regular movement the arc Mr, a quarter of the circumference of the circle MrVW, and that the whole of the circle EFGH is carried to Npt. The sun, regularly conveyed in the same direction, will describe the arc EF of the circumference of the circle EFGH. It will therefore be at point p and appear to us as being at s, and since it will have described the arc EF, a quarter of its own circle, it will appear to have traversed the arc ABs which is greater than a quarter of the zodiac, and to be rapidly moving from point A.

The center r will next describe the arc rV, a quarter of the circumference, and the circles Np and yV will be formed and the sun will travel through the arc pt, a quarter of the circumference, arriving at V while appearing to us to be at C, and seeming to have traversed the arc sC, less than a quarter of the zodiac, and to be slowly approaching point C. After point V has traversed the quarter VW of the circumference, its circle will be carried to XR and the sun, having described a quarter of the circumference, will be at point R, while appearing to be at point K and seeming to have described the arc CK, less than a quarter of the circumference, and to be slowly coming from point C.

Finally the center W, describing the arc WM, a quarter of the circumference,will re-establish the circle HR on EFGH, and the sun itself having described a similar arc XR will return to E and will appear to be at A. Then also it will seem to have described an arc KDA of zodiac, larger than a quarter of the circumference, and to be hastening to arrive at A. It is therefore evident that in its move-

[24]See III, XXVI.

ment it will appear to have a greater speed in Gemini and a slower speed in Sagittarius. This is, however, the contrary of what is observed. While the solar circle is carried on the circumference of the concentric circle MRVW in an inverse direction to that of the universe, the sun therefore cannot move on the epicyle in the same direction as this circle and in an inverse direction to the universe.

It remains to examine the case in which the epicycle has a movement contrary to that of the universe, and the sun moves on it in the same direction as the fixed stars. In this way the phenomena will be explained. (saved.)

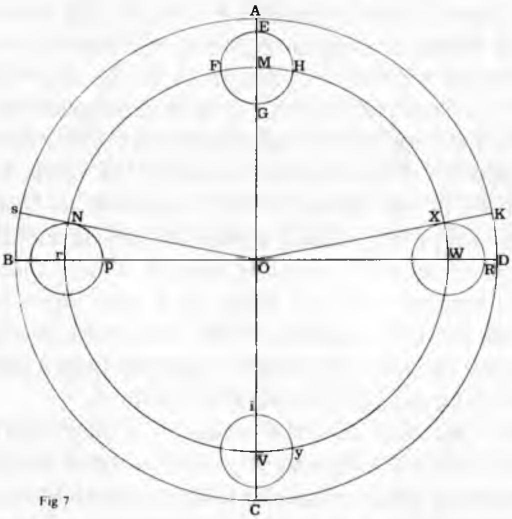


Fig 7

Indeed, let us suppose that the center of the epicycle describes the arc Mr, a quarter of the circumference of the concentric circle, and that it carries the epicycle with it to Np. The sun will have to describe the arc similar to EH on the epicycle, and will therefore be at N and will appear to us to be at s, having travelled through an arc equal to a quarter of its own circle. But on the zodiac it will seem to have travelled the smaller arc As at a low speed starting from point A. (Fig. 7)

Then the center r will describe a quarter of the circumference rV and the sun will describe the similar arc Np of the epicycle. It will then be at i and appear to be at C. It would seem to have travelled at an increasing speed towards C, the arc of the zodiac sBC, greater than a quarter of the circumference. If V were transported to W, the arc VW being a quarter of the circumference, and if the circle iy

were applied on the circle XR, the sun, in describing the arc Vy similar to the preceding one, will be at X, and will appear to be at K. It would seem to have travelled the arc CDK of the zodiac, larger than a quarter of the circumference, and to have passed rapidly from C to D.

If the center travels the remaining arc WM, the epicycle XR will return EFGH and the sun, describing the similar remaining arc XR will be reestablished at E. It will appear to be at A and will seem to have travelled the arc KA smaller than a quarter of the circumference, and to be slowly approaching A. In this way, following this hypothesis, all the phenomena are explained, for the sun will appear to move more slowly and to be smaller towards $5^1/_2$ degrees of Gemini and to move more quickly and be larger towards the same degree of Sagittarius. This in conformity with the phenomena, for if it passes from point E to point H while the center of the circle itself passes from M to r having a movement contrary (to that of its own circle) . . .

In moving to p during the time that the epicycle passes from r to V, the sun, which goes in the same direction as the epicycle, appears to advance on the zodiac in a movement which is in some way coincident with its own. Similarly, transported from V to y during the time that the epicycle passes from V to W, it will appear to increase speed in the zodiac, as if overtaking its own circle. But on the other hand, in passing from X to R while the epicycle passes from W to M, the sun, transported in a contrary direction to that of the movement of its own circle, will appear to accomplish its path on the zodiac slowly.

One can find the size of the epicycle and the relationship of the distance between the centers to the diameter EG of the epicycle EFGH. This relationship is the inverse of the preceding[25] since it is equal to the relationship of 24 to 1, obtained by the consideration of the distances and sizes. The greatest distance from the sun to the earth is OE, the smallest distance is OV and the difference between these two distances is equal to the diameter of the epicycle. Such is the explanation using the epicycle, the circle EFGH of the planet moving on a concentric circle which is MrVW.

In this way Adrastus shows that the phenomena are explained by the two hypotheses, that of the eccentric circles and that of the epicycle. Hipparchus made the remark that the reason why the

[25]Cf. XXVI. (1).

same phenomena follow from such different hypotheses, that of eccentric circles and that of concentric circles and of epicycles, is worthy of the attention of the mathematician. Adrastus has shown that the hypothesis of the eccentric circle is a consequence of that of the epicycle; but I say further that, the hypothesis of the epicycle is also a consequence of that of the eccentric circle.

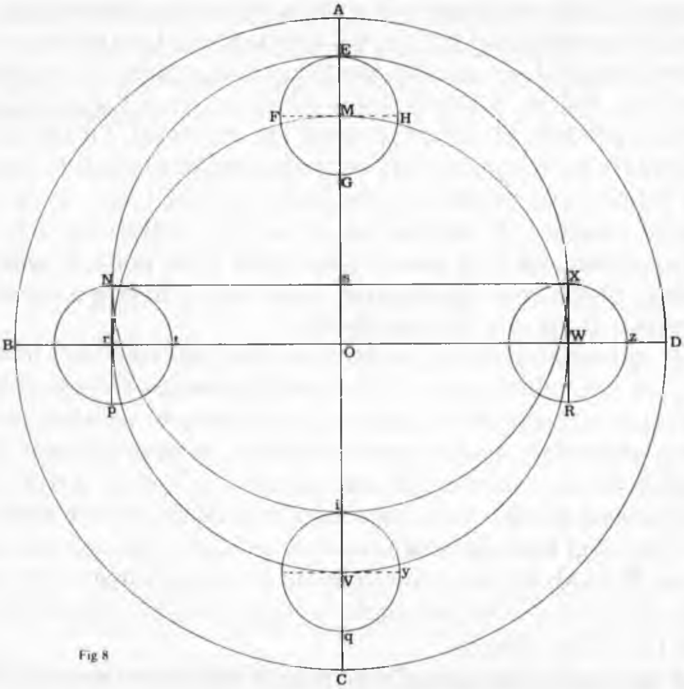


Fig 8

Let us take the zodiac ABCD with O as the center of the universe, and EFGH the epicycle of the sun with M as its center. Let us describe from the center O, with the radius OM the circle MrVW. I assert that if the center M uniformly travels the circumference of the homocentric circle MrVW in a movement contrary to that of the universe, and carrying with it the epicycle, it will happen that the sun, travelling the epicycle EHGF at the same time in a uniform movement in the same direction as the universe, will also describe the eccentric circle equal to the concentric circle MrVW. Let us then draw the diameters of the zodiac AC and BD, perpendicular to each other, in such a way that the point A is on the $5\frac{1}{2}$ degree of Gemini and C is on the same degree of Sagittarius, and from the centers r, V, and W, draw the circles Npt, iqy and XRz perpendicu-

lar to the diameter BD. And finally let us draw the straight line NX.

The lines NX and rW are equal and parallel to each other. Lines Ns and sX are therefore respectively equal to the lines rO and OW which are the radii of the circle MrVW. And since line Os is equal to rN, it will also be equal to both line iV and ME. But Ov = OM, therefore it also equals sE and Os = iV and line Oi is common. Therefore si = OV. Both line Es and line si are therefore equal to the radius of the arc VW. But it has been shown that each of the lines Ns and sX is equal to the radius of the circle, therefore the four lines sE, sN, si, sX are equal and perpendicular to each other. Thus the circle described from the center s with a radius equal to one of these lines will pass through points ENiX and be equal to the circle MrVW. It will be divided into four equal parts by the diameters Ei and NX. Let us describe this circle and suppose that it be ENiX. It will be eccentric, and the point which is projected at A, at 5½ degrees of Gemini, will be the furthest point from earth, and the point which is projected at C, at 5½ degrees of Saggitarius, will be the nearest.

I assert that the sun, moving as has been supposed, on the epicycle EFGH, will naturally describe the eccentric circle ENiX. In fact, the center of the epicycle described the arc Mr, a quarter of the circumference, and the sun at the same time will describe the similar arc EH of the epicycle and will arrive at N and then from E to N, having travelled through the quarter En of the eccentric circle. The center once more describes the quarter of the circumference, rV and the sun travels the similar arc Nt of the epicycle, therefore it will be at i and consequently describe the similar arc Ni of the eccentric circle. Similarly, during the time that point V describes arc VW, the sun will travel the similar arc iy of the epicycle, and will then be at X, having consequently described the similar arc iX of the eccentric circle. Finally, while point W travels the arc WM, the sun having described the arc Xz will return to E. It will therefore describe at the same time the remaining similar arc XE of the eccentric circle. Thus in travelling the whole epicycle uniformly while the epicycle is carried on the concentric circle, the sun describes an eccentric circle; this is what it was necessary to prove.[26]

[26]Assuming that the sun uniformly describes the epicycle in the direction of the diurnal movement, while the center of the epicycle uniformly describes the concentric circle in the opposite direction, Theon demonstrates that the sun is found on the eccentric circle at points E, N, i, X; but he does not demonstrate that the sun is on the eccentric circle at the intermediary points.

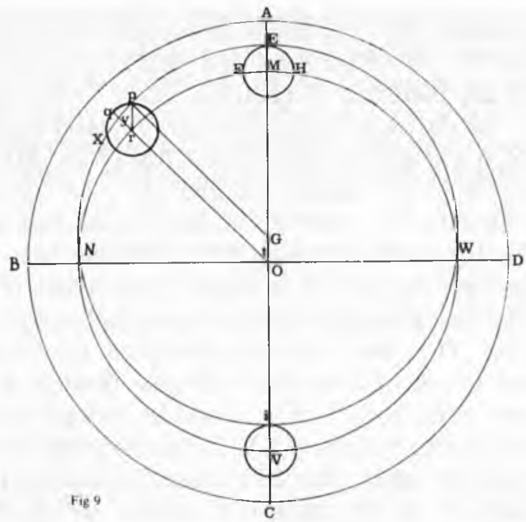


Fig 9

The same proposition is demonstrated in this manner: we have the zodiac ABCD and the solar epicycle EFH having its center on the circumference of the circle MrVW which is concentric around the center O of the universe. We also have point E, the point furthest from the earth, at 5 $^1/_2$ degrees of Gemini. I propose that the epicycle HE, being carried on the circumference of the circle MrVW by a uniform movement which is opposite to that of the universe, and the sun, travelling the epicycle EFH at the same time in a uniform movement which is opposite to that of the epicycle and consequently in the same direction as the universe, will then describe an eccentric circle equal to the concentric circle MrVW.

Let us now suppose that the center M has described any arc Mr, and that the epicycle has arrived at pqX. The sun, starting from point E, that is to say from point q, will have described at the same time arc qp, similar to arc Mr. Let us take the line OG, equal to the radius ME and draw the lines Gp and Oq. Since the arc qp is similar to the arc rM, the angle y is equal to the angle t. [27] Thus line pr is parallel to OG, and is also equal to it; the line pG is then equal and parallel to line rO. Now line Or is equal to line GE. Thus line Gp is equal to line GE. Therefore the circle described from the center G with radius GE, will pass through p and will be equal to the circle MrVW.

Let us describe the circle EpNiW (from the center G, with Gp = GE for radius); this circle will be eccentric. Since pG is parallel to

[27]Theon designates the angle qrp by y and the angle rOM by t.

qO, angle y will be equal to angle t, that is to say to pGE, the arc Ep is therefore similar to arc pq (of the epicycle pqX). The sun, starting from point E, will consequently describe the similar arc Ep of the eccentric circle. One could likewise demonstrate that it is always thus, so that the sun, having travelled the entire epicycle, propelling itself on a concentric circle, describes also a whole eccentric circle. This is what had to be demonstrated.

The converse proposition can also be demonstrated. Let us again take the zodiac ABCD whose diameter is AC and whose center is O, and again the eccentric circle of the sun ENiW, point E being furthest from the center of the earth, under 5 $^1/_2$ degrees of Gemini, and its center G on the line AO. [28] Let us describe from the center O, with the radius GE, the circle MrVW, and from the center M with the radius ME, the circle EFH. It is clear that this will be the same as the epicycle. I propose then that the sun uniformly describes the circumference ENiW of the eccentric circle, and will consequently also describe the epicycle EFH carried uniformly at the same time on the concentric circle MrVW.

Let us indeed suppose that the sun has described any arc Ep of the eccentric circle. Let us draw line pG and its parallel Oq. We take rq equal to OG and draw pr. Since the lines Gp and Or are equal and parallel, lines GO, and pr will also be equal and parallel. But we have OG = ME, therefore rq = rp, thus the circle described from the center r with radius rq will pass through point p and will be the same as the epicycle EFH. Let us describe this circle pqX. Because of the parallelism of the lines (rp and OG) the angles t and y are equal But in circles, similar arcs correspond to equal angles, and in equal circles, equal arcs correspond to equal angles, whether these angles be at the center or the circumference. Therefore, the arcs qp, Ep and Mr are equal.

Therefore, in the same time as the sun travels the arc Ep of the eccentric circle, the center M of the epicycle, describing the arc Mr, will carry the epicycle EFH to pqx, and the sun having travelled the arc Ep of the eccentric circle starting from point E, that is to say from point q, will describe the similar arc qp of the epicycle. It can be demonstrated that it is thus for the whole movement. Therefore, in travelling the entire eccentric circle, the sun also describes the entire epicycle. This is what it was necessary to demonstrate.

XXVII. The same demonstrations can be applied to the other planets. The sun appears to make all these movements, in both hy-

[28]Refer again to Fig. 9.

potheses, with regularity, for the times of its return to the same longitude, to the same latitude, and to the same distance which produces the irregularity called anomaly, are so little different from each other that most mathematicians regard them as equal $365\frac{1}{4}$ days. Thus, when one attentively considers the time of the return in longitude during which the sun travels the zodiac, going from one point back to the same point, from one solstice to the same solstice, or from one equinox to the same equinox, it is very close to the time noted above, so that at the end of four years, the return to a point at the same longitude occurs at the same hour.

As for the time of the anomaly after which the sun, at the point furthest from the earth, appears smallest and slowest in its movement towards the following zodiac signs, or after which, at the point closest to the earth, it appears to have the largest diameter and the greatest speed, it is close to $365\frac{1}{2}$ days, so that at the end of two years the sun appears to return to the same distance at the same hour. Finally, the time of its return in latitude, the time after which, starting from the extreme north or south point, it returns to the same point in such a way as to give the same shadow-lengths on the sundials, is $365\frac{1}{8}$ days. Consequently, it might be said that at the end of eight years, it will return at the same hour to the same point of latitude.

XXVIII. Regarding each of the other planets, we have said that their various times vary greatly, some are longer, some are shorter. The durations of the returns appears so much the more variable and changing in one hypothesis as in the next, that it is not in the same lapse of time that each planet travels its epicycle and the epicycle its concentric circle (in the zodiac): the movements are more rapid for some and slower for others by reason of the inequality of the circles, of the inequality of the distances from the center of the universe and of the differences of obliquity with respect to the circle of the middle of the zodiac signs, that is to say, of the differences of inclination and of position.

XXIX. As a result, it happens that for all the planets, the stations and the returns, whether towards the preceding zodiac signs or the following zodiac signs, do not occur in a similar manner. One observes the phenomenon for the five planets, but in a manner which is not absolutely similar. For the sun and the moon, it does not occur at all; indeed these two never appear to advance, nor to remain stationary, nor to retrogress. As we have said, the sun appears to be

carried on its own circle in the same time as the epicycle on the concentric circle, whereas the epicycle of the moon is carried more rapidly on the concentric circle through the zodiac, than it itself travels the epicycle.

XXX. It is clear that it matters little for interpreting the phenomena, whether one says, as it has been explained, that the planets move on circles or whether the circles which carry these stars move around their own centers. I understand that the concentric circles, carrying the centers of the epicycles, move around their own centers in the direction contrary to the universe, and that the epicycles carrying the planets also move around their own centers. Thus I understand that the concentric circle MNVW moves around O, (fig. 10) which is its own center and that of the universe, in the opposite direction from the universe. I understand in addition that the concentric circle carries on its circumference the center M of the epicycle EFGH and that this epicycle which carries the planet to point E, turns around the center M in the same direction as the universe, if it is a question of the sun and the moon, or in the opposite direction if one considers the other planets. In this way we can safely arrive at an explanation of these phenomena.

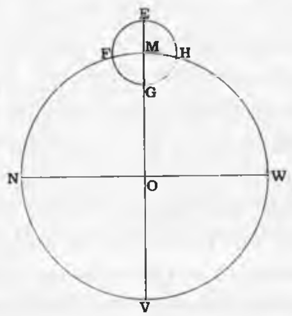


Fig 10

According to the other interpretation, we have the eccentric circle ENiW which has point H for its center. Considered in relation to the sun, this circle ENiW, moving uniformly in the space of one year around the center H, and carrying the sun fixed at point E, explains the phenomena, if the center H moves by itself, not in the opposite direction from the universe, but carried in the same direc-

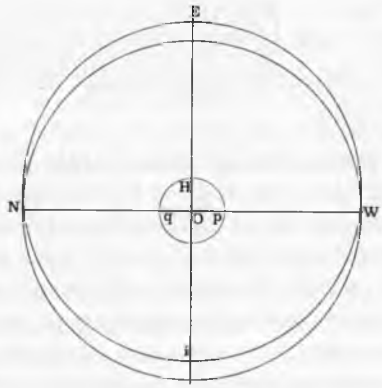


Fig 11

tion, and if each day it describes the circle Hqp equal to the circle in the other argument.

In this way, indeed, the sun will always present at the same respective places, the smallest and the mean distances from the earth. The largest, as has been said, occurs at $5^1/_2$ degrees of Gemini, and the smallest at the same degree of Sagittarius, and the middle at the same degree of Virgo and Pisces. In fact, the point E of the eccentric circle, where the sun is placed, seen under Gemini, in this position of the circle, is the farthest from the earth; but the circle turns around the center H, the point E, carried to where point i is now, will appear to us in Sagittarius at the smallest distance from the earth. Between these two extreme points it will be found at the middle of the distance in Virgo and Pisces.

As for the other planets, it is at every place in the zodiac that they can be at the greatest, the smallest or the mean distance from the earth, and that they can have their minimum, maximum or mean speed. From the center O of the universe and from the radius OH, let us imagine that one describes the circle Hpq, then that the concentric circle equal to the epicycle of the hypothesis turns around the center O of the universe and that it carries with it the center H of the eccentric circle in a movement contrary to the universe and in a determined time, and finally that the eccentric circle ENiW moves in a different time around its center H, carrying the fixed star on its circumference at point E. If one takes the correct and particular times of each planet, the phenomena will be explained.

All of this carries us too far away under the pretext of reconciling the arguments of mathematicians. Considering only the

phenomena and the planetary movements produced according to the course of things, after having observed them for a long time under favorable conditions in Babylonia, Chaldea and in Egypt, these mathematicians investigated with ardour the principles and the hypotheses which would explain the phenomena.[29] Thus they came to confirm the observed facts and to predict the coming phenomena, the Chaldeans with the aid of arithmetic methods, the Egyptians by graphic methods,[30] all by imperfect methods and without a sufficient science of natural laws, for it is necessary also to discuss the facts from the physical point of view. Those who have studied astronomy with the Greeks have tried to do so by utilizing their principles and their observations. Plato says so in the *Epinomis*, as we shall see a little further on, giving his own words.[31]

XXXI. Aristotle, in his treatise *On the Heavens*,[32] speaks a great deal of the stars in general and shows that they neither move across the tranquil ether nor with the ether in any independent or separate way, and that they neither turn nor revolve, but rather, that the numerous fixed stars are carried on one and the same sphere, the exterior sphere, and each planet is carried by several spheres. He again says in the XIth book[33] that according to Eudoxus and Callipus, the planets are put in motion with the aid of certain spheres. What in fact concurs with natural science is that the stars are neither carried in the same manner by certain circular or spiraling curves in a contrary movement to that of the universe, nor do these circles rotate around their centers, in carrying the various stars fixed to their circumferences, some moving in the same direction as the universe, some moving in the opposite direction. How could it indeed be that such bodies were attached to incorporeal circles?

According to the phenomena, the spheres of the fifth body[34] move in the depths of the sky; some are higher, some are less high, some are larger, others smaller, some are hollow, other full ones are inside them, and the planets which are fixed there in the manner of the stars, are carried by a simple movement but with unequal speeds depending on the location. Through an effect which is the

[29]Cf. Aristotle, treatise *On the Heavens* II, XII, 1 and *Meteorology*, I, VI, 9.

[30]Cf. Biot, *Journal des Savants*, 1850, p. 199.

[31]*Epinomis* p. 987 a.

[32]*Treatise On the Heavens*, II, 7.

[33]Aristotle, *Metaphysics*, XI, 1073 b.

[34]This fifth body is the ether. Cf. Aristotle, *Meteorology*, I, 3.

consequence of all these movements, they appear to move variously and describe certain eccentric circles; or else placed on other circles, they appear to describe spirals in consequence of which the mathematicians, misled by the retrograde movement, think that they were transformed.

As we see them carried each day by the motion of the universe from east to west and passing through the consecutive zodiac signs in their course along the obliquity of the zodiac, sometimes more to the north, sometimes more to the south, sometimes higher, sometimes lower, it follows that they appear more or less far from earth. Aristotle says that those before him supposed them each to be carried by several spheres.

Eudoxus says that the sun and the moon are supported on three spheres: the first is that of the fixed stars which rotate around the poles of the universe and forcibly draws all the others with it from the rising to the setting. The second moves around the axis perpendicular to the circle of the middle of the zodiac signs. It is by this sphere that each planet appears to execute a movement in longitude towards the zodiac signs that follow. The third rotates around the axis perpendicular to the circle, oblique to that of the middle of the zodiac signs. By the latter, each star appears to have its own movement in latitude, sometimes at a greater distance, sometimes at a smaller, sometimes more to the north, sometimes more to the south of the circle which passes through the middle of the zodiac. Each of the other planets is carried by four spheres, one of which produces the movement of the planet in height.

Aristotle says that Callippus added new spheres to the other planets, except to Saturn and Jupiter, that is to say two to the sun and the moon, and only one to each of the others. He also thinks that, if one wants to account for the phenomena, other lesser spheres than one of the spheres that carries the rotating ones, are necessary for each of the planets. Such is his opinion or that of the others (Eudoxus and Callippus). If one thought that it is natural that everything be carried in the same direction, one nevertheless saw the planets going in the opposite direction; also, one supposed that in the intervals of the conferring spheres (that is to say those carrying the planets), there are some evidently solid spheres which, by their own movement, cause the conferring spheres they contact to turn in the opposite direction, in the same way that with mechanical gears, wheels rotating around their axes, can, by their

own movement and by the aid of geared teeth, cause a turning and rotating in the opposite direction of the bodies adjacent and in contact.

XXXII. It is quite natural that all the spheres move in the same direction, carried along by the exterior sphere; but by an appropriate movement, because of the level which they occupy, their place and their size, some are carried more quickly, others less so and in the opposite direction, around their own axes oblique to that of the sphere of the stars. Thus the stars that they carry are brought along by the simple and regular movement of the spheres and it is only through an effect, which is the consequence of the movement of the spheres, that they appear to accomplish combined, irregular and varied movements; they describe several circles, some concentric, others eccentric or epicycles. For the understanding of what we say, it is time for a short elaboration on the figure which appears to us to be necessary for the construction of the spheres.

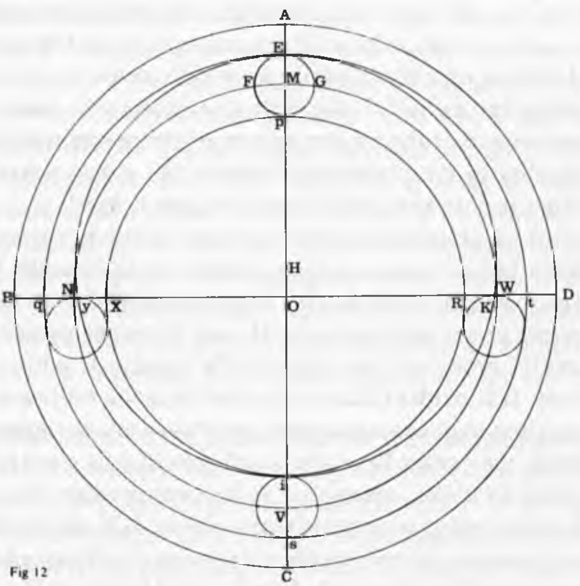


Fig 12

Let us take ABCD as the hollow sphere of the stars around the center O of the universe, AE, is its thickness, AC and BD its two (perpendicular) diameters. Let us suppose that ABCD is a large circle and that it passes through the middle of the zodiac. We have underneath the first, the hollow sphere Eqst, then pXiR of a planet

having the same center and Ep for thickness. Finally we have, within this thickness, the solid sphere EFpG carrying an errant star attached at point E. All are carried regularly in the same direction by the simple movements from east to west; and that which produces the movement of the planet in latitude alone turns in the opposite direction or in the same direction provided that it remains behind due to its slowness, for the two hypotheses account for the phenomena.

And now, it can be seen that the sphere of the stars turns around the axis perpendicular to the plane of the equatorial circle, and that the hollow sphere of the planet turns around the axis perpendicular to the circle producing the movement in latitude, and oblique to that which passes through the middle of the zodiac. Further the sphere of the stars turns very rapidly; the hollow sphere of the planet turns more slowly and in the opposite direction, in such a way that in a determined time, it has travelled the entire sphere of the stars in this opposite direction, or it is left behind as others would have it — we have said elsewhere that this is the most likely opinion — and it carries the solid sphere supporting the errant star. The solid sphere, turning regularly around its own axis, comes back to the same point, carried in the same direction as the starry sphere, in the same time that the hollow sphere of the planet will have travelled, moving in the opposite direction, the entire sphere of the stars, or that it will have left it more or less behind.

Let us first of all suppose that it returns in the same amount of time. Taking M for the center of the sphere, let us describe from the center O the circle MNVW having the radius OM. Let us divide the line Ei in two equal parts at point H, and from the center H, with the radius HE, describe the circle ENVW, eccentric with regard to the universe. It is evident that in the time that the hollow sphere of the planet, carrying the solid sphere, will have left the sphere of the stars behind, the center M of the solid sphere will travel the concentric circle MNVW, appearing to be conveyed in the opposite direction and carrying along this solid sphere. It is also evident that the planet placed at point E on the solid sphere will describe (in the same amount of time) the circle EGpF which becomes the epicycle of the concentric circle MNVW and turns in the same direction as the universe. It will consequently also describe the eccentric circle ENiW, equal to the concentric circle, travelling in it in the direction contrary to that of the universe.

It will therefore appear to observers at O to describe the zodiac ABCD, advancing towards the following zodiac signs in the opposite direction from the movement of the universe. It will also appear to move in latitude and in proportion to the inclination of its plane on the circle which passes through the middle of the zodiac signs, the axes of these spheres being respectively perpendicular to these planes. It is always at the same place that it will be the furthest from earth and that it will appear to move the most slowly: this is at point A of the zodiac, the center of the solid sphere being at point M of the line AO, and the planet itself being at the point E. At the opposite point it will always be least far from the earth and will appear to move most rapidly: this is at point C of the zodiac. The hollow sphere turning in the opposite direction, the center of the solid sphere will be at point V of the line OC and the planet itself will be seen at point C, that is to say, it will be at point i.

It will have mean distances and mean movements at two places: when it is at the points which divide the epicycle EFpG and the concentric circle MNVW into two equal parts. These points are F and G which, because of the changing of the spheres into the opposite direction, or because of their lesser movement, are the same as N and W which divide the eccentric circle ENiW and the concentric circle MNVW into two equal parts and appear in the zodiac between the points A and C at B and D, that is to say at y and K. All this is apparent for the sun, since the times of its returns, as far as our senses can perceive them, are found to be equal to one another or nearly so — I speak of the duration of its returns to the same longitude, to the same latitude, and to the same distance — the similar points of the two spheres are always found, through similar movements, at the same places and appear in the same zodiac signs.

Such a movement of the planets and the spheres is naturally regular, simple, and orderly, but it is oblique to the zodiac and, because of its slowness the planet appears to be left behind by the sphere of the fixed stars. A single sphere moves in the opposite direction, it is this which bears the solid sphere called the epicyle. However, the motion appears varied, complex and unequal. It is produced towards the following zodiac signs, either actually or by consequence of a slower displacement. It appears oblique to the zodiac and because of the rotation of the solid sphere around its own axis, the planet appears sometimes further away and consequently

slower, and sometimes nearer and consequently animated by a greater speed. In a word, the movement appears unequal, it is made following the epicyle and then it appears to be made following the eccentric. It evidently conforms to reason that there be an accord between the two hypotheses of the mathematicians on the motions of the stars, that of the epicycle, and that of the eccentric circle: both accord *by chance* with what is in conformity with the nature of things, which was the object of Hipparchus's admiration, particularly for the sun, since the motion of its spheres is accomplished in exactly equal times.

For the other planets there is not the same exactitude, because the solid sphere of the planet does not return in the same time to the same position. The hollow sphere remains behind that of the stars or goes in the opposite direction more or less rapidly, so that their similar movements, although accomplished on similar points of the spheres, are not always made at the same places, but ceaselessly changing place, the obliquity of the spheres not occurring at the same latitude, and the times of the returns to the same longitude, latitude and distance being unequal and variable, and the largest, the smallest and the mean distances, and likewise the variable speeds will be produced in all the signs of the zodiac, sometimes at one point, sometimes at another.

In addition, the similar movements appear, as we have said, to change place, although they occur at the same points of the spheres, the planets in their movements *by chance* do not even appear to describe circles, but spirals. It is necessary, therefore, to believe that each planet has its own hollow sphere which carries a solid sphere in its thickness, and that the solid sphere in turn carries the star on its surface.

XXXIII. As for the Sun, Venus and Hermes, it is possible that each of these stars has two spheres of its own, and that the hollow spheres of the three stars, animated at the same speed, travel in the same time, in the contrary direction, the sphere of the fixed stars and that of the solid spheres have their centers on the same straight line, the sphere of the sun being the smallest, that of Hermes being larger, and that of Venus being still larger.

It could also be possible that there be a single hollow sphere common to the three stars and that the three solid spheres in its thickness have one and the same center. The smallest would be the truly solid sphere of the sun, around which would be that of

Hermes, then that of Venus would come after encircling the other two and filling up the whole thickness of the common hollow sphere. This is why these three stars are left behind on the zodiac, or execute a motion in longitude in the opposite direction to the diurnal movement and at the same speed without having other similar motion. They always appear neighboring, mutually passing each other and eclipsing each other. Hermes has a distance from the sun of at most twenty degrees on one or the other side of it at the setting and rising, and Venus of 50 degrees at most. It will be understood that this position and this order are all the more true as the sun, essentially hot, is the world's hearth, and for the world, as with an animal, it is so to say the heart of the universe, because of its movement, its size and the common course of the surrounding stars.

For in animate beings, the life center, that is to say, of the animal, as a living thing, is different from the center of the body. For example, for us who are, as we have said, both humans and alive, the center of life is in the always warm and always moving heart, and because of this the source of all the faculties of the soul, the cause of life and of all movement from one place to another, the source of our desires, of our imagination and intelligence. The center of our body is different: it is situated towards the navel.

Likewise, if we judge the largest, most worthy and most divine things, like the smallest, fortuitous and mortal, the center of the body of the universal world will be the cold and immobile earth, but the center of the world, judging it as world and as a living thing, will be in the sun which is in some way the heart of the universe and in which it is said the soul of the world took birth in order to penetrate and extend itself to its furthest ends.

XXXIV. It is clear then, that for the reasons explained, of the two hypotheses, each of which is a consequence of the other, that of the epicycle appears most common, most universal and most consistent with the nature of things. For the epicycle is a large circle of the solid sphere, the one which a planet describes in its movement on this sphere, whereas the eccentric circle differs entirely from the circle which is in conformity with nature and is rather described *by chance*. Hipparchus, persuaded that the phenomenon was produced in this way, vaunts the hypothesis of the epicycle as his own and says that it is probable that all the celestial bodies are uniformly placed with respect to the center of

the world and that they are similarly united. But he himself, not knowing sufficient natural science, did not understand which is the true movement of the stars in accord with the nature of things, and which is the movement which is *by chance* and is but an appearance. He hypothesizes that the epicycle of each planet moves on the concentric circle and that the planet moves on the epicycle.

Plato appears also to prefer the hypothesis of the epicycle. He thinks that it is not of spheres, but of circles which carry the planets, as he indicates at the end of the *Republic* in imagining spindle-wheels inside one another. He uses these terms in common: he often says circles instead of spheres, and around the poles instead of around the axes.

[According to the appearances, says Aristotle, the spheres of the fifth body (the ether) move in the depths of the sky. Some are higher, others less so, some are larger, others smaller, some are hollow, with other full ones inside them, and the planets, which are fixed to them in the manner of the stars, are carried in a simple movement, but of unequal speeds according to the location. Through an effect which is the consequence of all these movements, they appear to move diversely and to describe certain eccentric circles; or else, placed on other circles, they appear to describe the spirals following which, mathematicians, deceived by the retrograde movement thought they were transformed into.]

XXXV. It is necessary to show how it is that several planets appear sometimes to advance, sometimes to be stationary, and sometimes in retrograde movement, for they appear to do all these things.

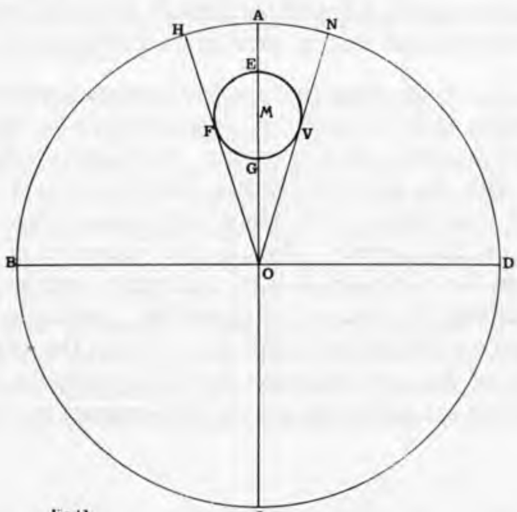


Fig 13

We have the zodiac ABCD around point O, the center of the universe, and EFG the epicycle of a planet. From the point which we observe, let us draw the tangents OFH, OVN to the epicycle and through the center M of the epicycle, the line OMEA. Since we see in a straight line, it is clear that the star F appears to us to be at H. then when it travels the arc FE, it will appear to us to describe the arc HA towards the preceding signs of the zodiac. During the time it approaches point F, it will again advance. Next, in approaching point V and in beginning to move away again, it will appear to be stationary and finally to make a retrograde motion. The stations, retrograde movements, and the movements forward and backward of each planet, are sometimes made in one zodiac sign, sometimes in another and in different parts of the zodiac signs, because the epicycle of each of them is always moving towards the following signs, whether this movement be real, or whether the epicycle is simply left behind.

THE MEAN DISTANCES OF THE PLANETS

XXXVI. It is useful for our subject to know what the average distance of a planet is, and what the displacement of the epicycle or of the eccentric circle is. If, in the epicycle hypothesis, we take the greatest distance from the star to the earth, such as OE, and then the smallest,

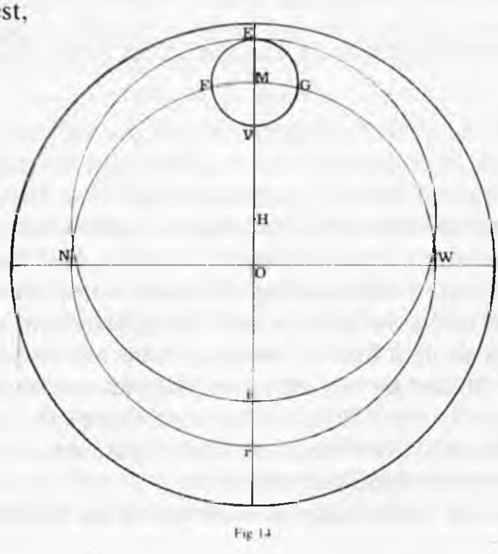


Fig 14

such as OV, as well as the distance between the largest and the smallest, that is to say, EV, and we find the middle, M, it is clear that the mean distance will be OM. If therefore, from the center O and with the radius OM, we describe the concentric circle MNrW, and from the center M with the radius ME we draw the epicycle EFVG, it is evident that the star carried on the epicycle will be furthest away from us at point E and least far at point V, and at a mean distance at the two points F and G on the intersection of the concentric circle and the epicycle, on the path of the epicycle.

For the hypothesis of the eccentric circles, let us take the eccentric circle ENrW, the center of which is H and O is the center of the universe. Let us take the line of the centers OH and extend it on both sides. If we describe, from the center O the circle MNrW equal to the eccentric circle, it is clear that it is the concentric circle on which the epicycle of the other hypothesis is carried, described from the center M with the radius ME. When the planet carried by the eccentric circle arrives at E, in whatever place this happens, it will be furthest away from us, it will be least far at point i, and the mean distances will be at the points of intersection N and W of the eccentric and concentric circles in whatever place these points fall through the displacement of the eccentric circle. It is evident that there is an agreement between the two hypotheses: the greatest, the smallest and the mean distances are the same.

CONJUNCTIONS, OCCULTATIONS AND ECLIPSES

XXXVII. To fulfil the requirement of our subject, it remains to speak briefly of conjunctions and occultations, disappearances and eclipses. Since we naturally see in a straight line, that the sphere of the stars is highest and that the planetary spheres are placed below in the order that we have indicated, it is clear that the moon being the planet closest approaching the earth, can pass in front of all the planets and several stars and hides them from us, when it is placed in a straight line between our view and these stars, and it cannot be hidden by any of them. The sun can be hidden by the moon, and can itself hide all the stars except the moon, first in approaching and drowning them in its light, then by being placed directly between them and ourselves.

Hermes and Venus hide the stars which are beyond them when

they are similarly placed in a straight line between them and ourselves. They even appear to eclipse each other when one of the two planets is higher than the other or because of their sizes, or the obliquity or position of their circles. The fact is not easy to observe, because the two planets turn around the sun, and Hermes, in particular, which is only a small star, is the sun's neighbor and is drowned out by it and is rarely apparent. Mars sometimes eclipses the two planets which are above it, and Jupiter can eclipse Saturn. Each planet eclipses in addition the stars underneath which it passes in its course.

ECLIPSES OF THE SUN AND MOON

XXXVIII. The moon disappears when, diametrically opposite the sun, it enters into the shadow of the earth. This does not happen every month, and the sun is not eclipsed at every junction of the moon or neomenia, in the same way that the moon is not eclipsed at all full moons, because their circles are perceptibly inclined to one another. The circle of the sun appears conveyed as we have said [35] under that which passes through the middle of the zodiac signs on which it is a little inclined, for it is separated from it by a half degree on each side; and the circle of the moon has an obliquity of ten degrees in latitude, as Hipparchus has found, or of twelve degrees as most mathematicians think, so that it appears to be separated by five or six degrees to the north or to the south of the circle which passes through the middle of the zodiac signs.

If we suppose the planes of the two circles, the solar and the lunar, to be extended, their common intersection will be a straight line which touches the center of the two circles. This line, in some way, will be their common diameter. The extreme points where the circles appear to be cut are called the nodes, one ascending, the other descending. They are carried towards the following signs of the zodiac. If the conjunction of the sun and moon occurs near the nodes, the two stars appear close to one another and to our eyes the moon will hide the sun which will be more fully eclipsed the more the moon covers it. But if the monthly conjunction does not occur near a node, the longitude counted on the zodiac being the same for the two stars, but the latitude being different, one of the two stars will appear further north, the other further south, and the sun not being hidden, will not seem to be eclipsed.

[35] See III, XII.

XXXIX. Let us now see what evidently happens for the moon. It is eclipsed, as we have often said, when it enters into the shadow of the earth. Let us show how it happens that the eclipse does not take place each month. The luminous rays which extend in a straight line, envelop an obscure region; if two spherical bodies, one luminous and the other lit by the first, are equal, the shadow produced is a cylinder that extends to infinity. Let us take, for example, AB, the luminous body, and CD the illuminated body, and suppose them to be equal and spherical. The rays of light such as AC and BD extend in a straight line, therefore the diameters AB and CD being equal and perpendicular to the tangents ACE and BDF, it is clear that these rays will be parallel and that the lines CE and DF, extended indefinitely, will not meet each other. As this occurs at all the points it is evident that the sphere CD will produce an indefinite cylindrical shadow.

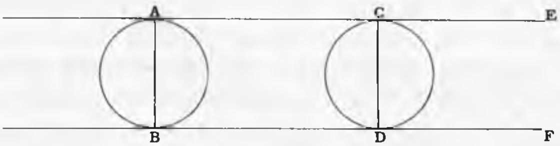


Fig 15

If, on the contrary, the luminous body is smaller, such as GO and the illuminated body is larger, such as XN, the shadow XMNV will have the form of an indefinite truncated cone, for the diameter XN being larger than the diameter GO, the luminous rays XM and NV, extended indefinitely, will move further and further apart from one another, and it will be thus in every case (total eclipse).

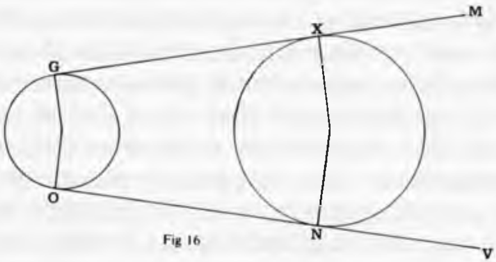


Fig 16

If the luminous body is larger, such as Wr and the illuminated

body smaller, such as pq and both are spherical, it is clear that the shadow of the body pq, that is to say, pqs, will have the form of a cone and will be finite, for the rays Wp and rq, extended in a straight line, will meet at point s, since the diameter pq is smaller than the diameter Wr. This phenomenon will be produced from every point.

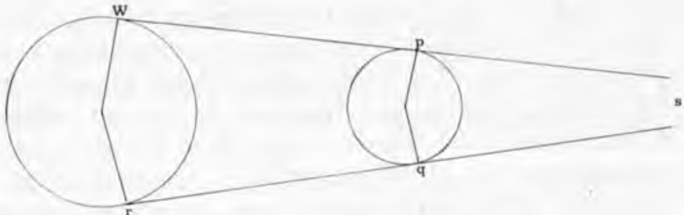


Fig 17

Through the consideration of the distances and of the diameters of the sun and moon, Hipparchus shows that the volume of the sun contains that of the earth about 1880 times, and that the volume of the earth contains that of the moon more than 27 times, and that the sun is much further away than the moon. It is then evident that the shadow of the earth will have the form of a cone, that it will extend following a common diameter of the sun and of the earth (that is to say following the line which joins their centers), and that the diameter of the moon, even at its maximum, is less than the width of the shadow projected by the earth. When the sun is at a node and the moon at another node, the sun, the earth, and the moon being in a straight line, the moon necessarily enters into the shadow of the earth, and as it is smaller and has no brilliancy by itself, it becomes invisible, and it is said that it is eclipsed.

When the centers of the sun, the earth, and the moon are exactly placed following a diametrical line, that is following the same straight line as we have described it, the moon penetrating to the middle of the shadow, there is a total eclipse. When the three centers are not totally in a straight line, there is not always a total eclipse. But most often, at the time of the full moon, the sun and the moon do not pass through their nodes, and the moon will be further to the north or further to the south than the shadow of the earth. As it does not enter into the shadow cone it would be known that there could be no eclipse.

That is what Adrastus says. Dercyllides has not written on this

subject in any convenient order. Here, however, is what he indicates in the book in which he treats *The Spindles with which the Republic of Plato is Concerned.*

ASTRONOMICAL DISCOVERIES AND THEIR AUTHORS

XL. Eudemus, in his books *On Astronomy,* tells that Oenopides[36] found the first obliquity of the zodiac and recognized the existence of the great year. According to him, Thales was able to see that the eclipses of the sun and the returns of this star to the solstices do not always occur after the same lapse of time. Anaximander maintains that the earth is suspended in space and moves around the center of the world. Anaximenes has shown that the moon receives its light from the sun and the manner in which it is eclipsed. Others have added new discoveries to these: that the stars move around the immobile axis which passes through their poles, that the planets move around the axis perpendicular to the zodiac; and that the axis of the stars and that of the planets are separated from one another by the side of the pentadecagon, and consequently by an angle of 24 degrees.

ASTRONOMICAL HYPOTHESES

XLI. He then says that, just as in geometry and in music it is impossible, without making hypotheses, to deduce the consequences of the principles, also in astronomy it is convenient to establish hypotheses first in order to be able to speak of the movements of the planets. Before all else, he says, as everyone agrees, it is necessary especially to consider principles which should serve in the study of mathematics. The first is that the composition of the world is ordered and governed by one sole principle and that reality is found at the bottom of things which exist or seem to exist, and that one must not say that the world is infinite or our vision becomes lost, but that it has its limits.

The second principle is that the risings and the settings of the divine bodies are not made because these bodies light up and extinguish themselves successively; if their state were not eternal, there would be no order preserved in the universe. The third principle is that there are seven planets, neither more nor less, a truth

[36]The obliquity of the zodiac is attributed to Pythagoras but Oenopides claims it as his own discovery. (Toulis)

resulting from long observation. The fourth is the following: since it does not conform to reason that all things be in movement or that they all be in repose, but since some are in movement and others immobile, it is necessary to find out that which is necessarily at rest in the universe and that which is in movement. He adds that one should believe that the earth, hearth of the house of the gods, according to Plato [37] remains in repose and that the planets move with the whole celestial vault which envelops them. Next, he energetically rejects, as contrary to the bases of mathematics, the opinion of those who believe that the bodies which appear to be in movement are at rest and that the bodies, immobile by nature and by situation, are in movement.

He then says that the planets have a circular movement, regular and uniform, in longitude, in distance and in latitude...He judges it so, although we can be deceived on this point. This is why he believes that the different successive risings depend on a movement in longitude and he rejects the weak and facile reasons given by the seniors according to which the planets would be left behind. Putting aside anything disorderly and contrary to reason in such a movement, it is correct to believe, he says, that the planets are carried slowly by a movement contrary to that of the fixed stars, in such a way that the interior movement is produced by the exterior movement.

He does not think that it is necessary to take spirals or the similar lines of a meandering horse-path as the first cause of these movements. For these things come about by chance. The first cause of the spiral movement is the movement which occurs following the oblique circle of the zodiac. The spiral movement is in fact, adventitious and posterior, it results from the double movement of the planets. Therefore, the movement following the oblique circle must be regarded as first; the spiral movement is a consequence, it is not first.

In addition, he does not believe that the eccentric circles are the cause of the movement in depth. He thinks that all that moves in the sky is carried around a unique center of movement and of the world, in such a way that it is not by a consequence and not by an antecedent movement, as we have said above, that the planets describe the epicycles or the eccentric circles within the thickness of the concentric circles. For each sphere has a double surface, one side concave at the interior, the other convex at the exterior, in the interval between which the stars move following the epicycles and

[37] Cf. *Phaedrus*, 247 A.

concentric circles, in a movement which causes them to describe the eccentric circles as an apparent consequence.

He further says that, according to our impressions, the movements of the planets are irregular, but that in principle and in reality they are regular. The movement is simple and natural for all: there is but a small number of displacements on the spheres arranged with order. He reproaches those philosophers who, considering the stars as inanimate, add several other spheres to their spheres and their circles; thus Aristotle,[38] and among mathematicians Menaechmus and Callippus, have proposed different spheres and spirals. After having established all this, he thinks that the sky moves with all the stars around the immobile earth, following a very small number of circular, uniform, harmonious, concentric and independent movements. He shows, that, according to Plato, these hypotheses account for the appearances.

XLII. The stars move around the immobile axis which passes through the poles, and the planets around the axis perpendicular to the zodiacal circle. The two axes are separated by the value of the side of the pentadecagon (and consequently by an angle of 24 degrees). Indeed the zodiac being a great circle, divides the world into two equal parts. The circumference of the universe being divided into 360 degrees, the zodiacal circle separates it into 180 degrees on each side. The axis of the zodiac itself being perpendicular, also divides the 180 degrees into two equal parts. Now the zodiac extends obliquely from the parallel of winter to the parallel of summer, but 30 degrees are counted from the tropic of summer to the arctic circle, as Hipparchus teaches, and from the arctic circle to the pole of the sphere of the stars there are 36 degrees. In summing them up, 66 degrees are then counted from the tropic of summer to the pole of the fixed stars.

In order to complete the 90 degrees which extend up to the pole of the sphere of the planets, it is necessary to add 24 degrees to this sum, then the axis of the planets is perpendicular to the zodiac. There remain 12 degrees from the pole of the axis of the planets to the glacial arctic circle, for the entire arc of the zone is 36 degrees; if 24 degrees are subtracted from it, the remainder is 12. He agrees to add the 30 degrees comprising from the arctic circle to the tropic of summer, then the 24 degrees comprising from the tropic of summer to the equitorial circle, and again the 24 degrees comprising from the equitorial circle to the tropic of winter to which the

[38]Cf. *Metaphysics* 8, p. 1073b.

zodiac is tangent. But 24 degrees form a fifteenth of 360 degrees of the circumference of the universe, for 15 times 24 makes 360. We then have reason to say that the two axes, that of the stars and that of the planets, are separate from one another by the value of the side of the pentadecagon inscribed in (a great circle of) the sphere.

XLIII. It so happens that the planets describe spirals, that is to say in consequence of their two movements in opposite directions from one another. In fact, as they are carried by their own movement from the tropic of summer to the tropic of winter and reciprocally, going slowly, as they are rapidly pulled along each day in the opposite direction under the sphere of the stars, they do not pass in a straight line from one parallel to another, but are pulled around the sphere of the fixed stars. In other terms, in order to go from one point A to another point B on the zodiac, their motion does not only occur following a straight line of the zodiac, but it becomes at the same time circular around the sphere of the fixed stars, in such a way that in passing from one parallel to another they describe spirals similar to the tendrils of a vine; it is as though one were to wind a strap around a cylinder from one end to the other, such were the straps wound on the staffs of Sparta and on which the overseers[39] wrote their dispatches.

The planets describe still another spiral, but this one is not as might be drawn on a cylinder from one end to the other, but as could be drawn on a plane surface. Since from an infinite time they pass in a circle parallel to the other, and again to that of the first, and this without interruption and without end, if we suppose straight lines, arranged in an infinite number, representing the parallel circles and that the planets move on these parallels in the same direction as the sphere to the fixed stars, sometimes towards the tropic of winter, sometimes towards the tropic of summer, they will appear to us to describe a helix without end. Because of the incessant and continuous movement around the sphere on the parallel circles, the path-way travelled will be similar to that which

[39]The overseers were a body of five lords in ancient Sparta who had jurisdiction over even the king. They used a wooden staff of exact dimensions around which a leather strap or paper strip was wound in the mode of a spiral. Then they wrote across the length of the staff whatever message they wanted dispatched to their generals at war in far away lands. After the writing was dry, they unwound the strap and sent the written message by special dispatchers to the general concerned who then read it after winding it around his staff of exact dimensions as the one at headquarters. It was a way of coding and decoding military and other classified data because to someone who was not in possession of this wooden staff, any message so written was meaningless. (Toulis)

would be made following straight lines extended to infinity, as the figures show.[40] It so happens, then, that the planets describe two spirals, one is around a cylinder, the other as on a plane surface.

XLIV. All of this is very necessary and very useful for reading the works of Plato. However, we have said that we were going to consider instrumental music, mathematical music and the harmony of the spheres [41] and that we would report all that necessarily exists of harmony in the world, and after which look at astronomy, for Plato assigns the fifth step in mathematics to this music of the spheres, after arithmetic, stereometry and astronomy [42] — we are going therefore to show summarily what Thrasyllus exposes on this subject at the same time as our own previous work.

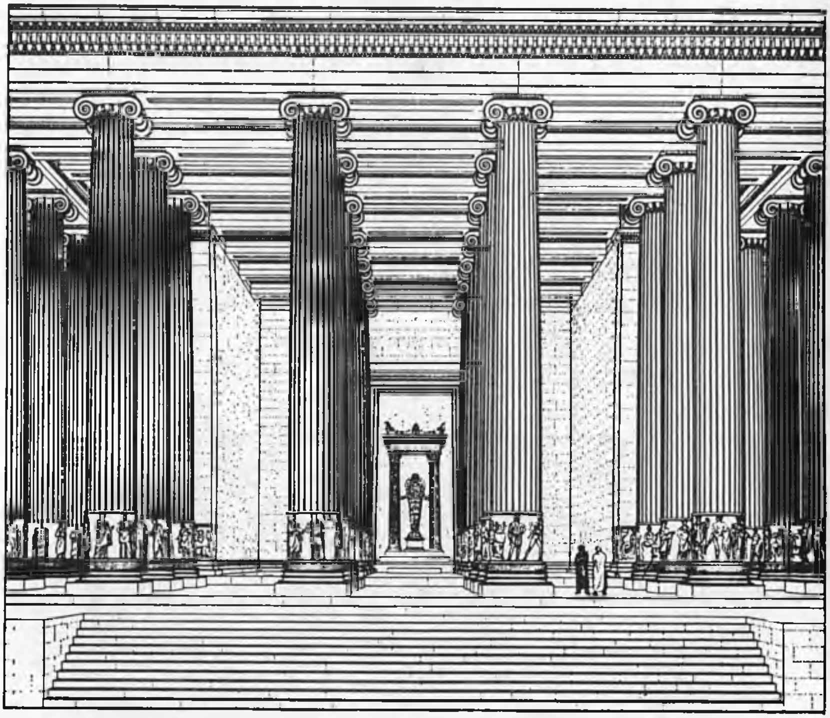
END OF THE TRANSLATION
OF THE WORKS OF
THEON OF SMYRNA
AS THEY HAVE COME DOWN TO US.

[40]These figures are missing from the manuscript.

[41]See I, 2 and II, 1.

[42] *Republic VII, p. 530 D.*

NOTES



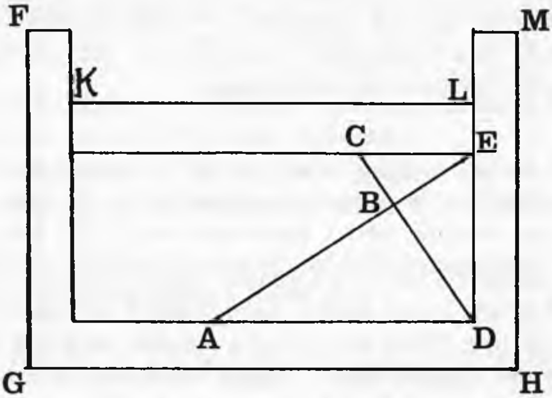
NOTES — (by J. Dupuis, 1892)

Note I. - *Problem of the duplication of the cube, Mechanical solution of Plato* (Introduction p. 5)

The problem of the duplication of the altar, with the condition that the new altar be similar to the first, goes back to the duplication of the cube of one edge. Hippocrates of Chio found that if two continuous proportional means x and y are inserts between the side (a) of a cube and the double (2a) of this side, the first mean x is the side of the double cube. We have, in fact, by definition:

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{2a} \text{ from which follows } \frac{a^3}{x^3} = \frac{axy}{2axy} = \frac{1}{2} \text{ and } x^2 = 2a^3$$ [1]

Plato first resolved the problem of the two proportional means. He employed an instrument formed of two rulers, KL and GH, of which one moveable one, parallel to the other fixed one, glided between the grooves of the two uprights FG, and MH perpendicularly attached to it.



Taking (a) and (b) as the two lines between which the two proportional means are to be inserted, we draw two perpendicular lines AE and CD on which we take, starting from their point of intersec-

[1]The ancient geometers were not able to avail themselves of the resources of algebra which they did not know; but proportions, which they managed with great facility, even though they had no particular notation, furnished them with very simple and very ingenious procedures of calculation. In combining proportions by means of multiplication, of division. . . and in simplifying the relationships of the final proportion, they arrived at conserving only one unknown in questions which called for several.

tion, AB = a and BC = b. Then we apply the instrument on the figure in such a way that the edge of one ruler passes through point A, and the edge of the other through point C. We then separate the moving ruler more or less from the fixed one, and at the same time turn the instrument on the surface of the figure until the edges of the two rulers again pass through points A and C; the extensions of the lines AB and BC pass at the same time through the apexes of the rectangle which forms the instrument.

The two triangles ADE, and CDE being right triangles, the height of each of them is the proportional mean between the segments of the hypotenuse, and we have:

$$\frac{AB}{BD} = \frac{BD}{BE} = \frac{BE}{BC}$$

Thus BD and BE are two proportional means between AB and BC, that is to say, between a and b.

This solution of Plato is mechanical, since it requires the utilization of an instrument other than the ruler and the compass. It has been transmitted to us by Eutocios of Ascalon, a geometer of the VIth century, in a commentary on book II of the treatise *On the Sphere and the Cylinder* by Archimedes.[2]

Note II. — *On the sophism, "One, as One, is without parts and indivisible"* (I, III). — *Problem of Achilles and the Tortoise.*

The reasoning of Theon is a sophism. I have a tangible object, he says, and I divide it into several parts which I successively throw away one by one. There will come a moment in which there remains only *one* tangible object. I again divide this *one* tangible object into several parts which I throw away one by one, until there remains no more than one object. Continuing in this way, I always arrive at *one*, therefore *one,* as *one,* is without parts and indivisible.

One of the most celebrated sophisms is that of Zeno of Elea who lived in the Vth century before our era. It is called *Achilles.* It is stated as follows:

[2] Cf. *Archimedis quae supersunt omnia cum Eutocii Ascalonitae commentariis ex recensione Josephi Torelli Veronensis....Oxonii* MDCCXCII, in-fol. p. 135

Achilles travels ten times faster than a tortoise which is one stade ahead of him. Will he catch up with it and at what distance?[3]

Zeno claimed that Achilles would never reach the tortoise because, he said, while Achilles was travelling the stade which separated him from the tortoise, the latter advanced 0.1 of a stade; and while Achilles travels this tenth, the tortoise, who travels ten times slower, will advance by 0.01 of a stade; and while Achilles travels this hundredth, the tortoise will advance by 0.001 and so forth. Therefore an infinite number of moments will pass before he reaches it, and so Achilles will never catch up with the tortoise.

This comes back to the affirmation, says Aristotle, "that the slower while he is travelling, will never be reached by the faster considering that the pursuer must, of necessity, first pass by the point from which the one who flees his pursuit started, and thus the slower will constantly preserve a certain advance." *Lesson in Physics,* VI, IX, 4.

The error of Zeno is evident, for Achilles reaches the tortoise at a distance from its point of departure, equal to 1 stade and $^1/_9$ th or $^{10}/_9$ ths of a stade. In fact, during the time he travels these $^{10}/_9$ ths of a stade, the tortoise, who goes ten times slower, travels $^1/_9$ th of a stade; now the space travelled by Achilles is then equal to the space travelled in the same time by the tortoise, plus the space which separated them, therefore they meet each other.

Zeno did not see that the sum of the spaces travelled during the infinite number of successive moments of the movement of Achilles and the tortoise represents a finite distance, and that, in the case of the uniform movement, the infinite number of these successive moments represents a finite time. [4]

Note III — *On unequilateral numbers* (I, XVI).

Let us take $|(n - 1)n = n^2|$ and $n(n + 1) = n^2 + n$, two successive unequilaterals. The square contained between $n^2 - n$ and $n^2 + n$ is n^2. Now n^2 is the arithmetic mean between $n^2 - n$ and $n^2 + n$, and the arithmetic mean between two numbers is larger than their geometric mean. Therefore, as Theon verifies, the square contained

[3] The stadia has the value of 185 metres. (606′ +)

[4] We do not insist upon it, but we bring to the attention of the *philosophic* reader the interesting work recently published M.G. Frontera, doctor of sciences, under the title: *Edude sur les arguments de Zènon d'Elèe contre le mouvement,* Paris, Hachette, 1891, br. in 8^e.

between two successive unequilaterals is not the geometric mean between these two numbers. But the geometric mean between two successive squares is an unequilateral. We have, in fact x, the geometric mean between two successive squares n^2 and $(n + 1)^2$, thus $x^2 = n^2 (n + 1)^2$, from which it follows that $x = n (n + 1)$, an unequilateral number, since the two factors differ by one unit.

Note IV. — *On Square Numbers.* (I, XX).

Every number being a multiple of 6 or a multiple of 6 plus 1, plus 2, plus 3, plus 4 or plus 5 is of the form $6n$, $6n \pm 1$, $6n \pm 2$, or $6n \pm 3$. Thus any square is of the form

$$36n^2 \quad 36n^2 \pm 12n + 1 \quad 36n^2 \pm 24n + 4 \quad \text{or } 36n^2 + 36n + 9.$$

1. If it is of the form $36n^2 \pm 24n + 4$, it is divisible by 4 and not by 3, but the subtraction of one unit gives the remainder $36n^2 \pm 24n + 3$, which is divisible by 3.
2. If it is of the form $36n^2 + 36n + 9$, it is divisible by 3 and not by 4, but the subtraction of one unit gives the remainder $36n^2 + 36n + 8$, which is divisible by 4.
3. If it is of the form $36n$, it is divisible by both 3 and 4 and consequently $36n^2 - 1$ is not so.
4. Finally, if it is of the form $36n \pm 12n + 1$, it is divisible neither by 3 nor by 4, but the subtraction of one unit gives the remainder $36n^2 \pm 12n$ which is divisible by both 3 and 4.

Note V. - *On Polygonal Numbers* (I, XIX - XXVII.)

We shall summarize this theory of polygonal numbers and add some explanations.

Taking d as the rate of a progressing by difference beginning with unity, the first terms will be

$$1,\ 1 + d,\ 1 + 2d,\ 1 + 3d,\ 1 + 4d,\ 1 + 5d,\ 1 + 6d,\ 1 + 7d\ldots$$

If we take the successive sums of the terms, starting from unity, we will obtain the corresponding numbers

$$1,\ 2 + d,\ 3 + 3d,\ 4 + 6d,\ 5 + 10d,\ 6 + 15d,\ 7 + 21d,\ 8 + 28d\ldots$$

The terms of the second series are called *polygonal numbers,* and those of the first series are called *gnomons.* If, in the two series, we give to d the successive values 1, 2, 3, 4, 5, 6...we will obtain the following gnomons and polygonal numbers:

d = 1, gnomons...	1	2	3	4	5	6	7	8	9	10	11	12
n. triangulars.....	1	3	6	10	15	21	28	36	45	55	66	78
d = 2, gnomons...	1	3	5	7	9	11	13	15	17	19	21	23
n. quadrangulars..	1	4	9	16	25	36	49	64	81	100	121	144
d = 3, gnomons...	1	4	7	10	13	16	19	22	25	28	31	34
n. pentagons	1	5	12	22	35	51	70	92	117	145	176	210
d = 4, gnomons...	1	5	9	13	17	21	25	29	33	37	41	45
n. hexagons	1	6	15	28	45	66	91	120	153	190	231	276
d = 5, gnomons...	1	6	11	16	21	26	31	36	41	46	51	56
n. heptagons	1	7	18	34	55	81	112	148	189	235	286	342
d = 6, gnomons...	1	7	13	19	25	31	37	43	49	55	61	67
n. octogons	1	8	21	40	65	96	133	176	225	280	341	408
d = 7, gnomons...	1	8	15	22	29	36	43	50	57	64	71	78
n. enneagons	1	9	24	46	75	111	154	204	261	325	396	474
d = 8, gnomons...	1	9	17	25	33	41	49	57	65	73	81	89
n. decagons	1	10	27	52	85	126	175	232	297	370	451	540
d = 9, gnomons...	1	10	19	28	37	46	55	64	73	82	91	100
n. endecagons	1	11	30	58	95	141	196	260	333	415	506	606
d = 10, gnomons..	1	11	21	31	41	51	61	71	81	91	101	111
n. dodecagons....	1	12	33	64	105	156	217	288	369	460	561	672

By designating the nth gnomon as K and the nth polygonal number by j, we have:

$$k = 1 + (n - 1)d$$

and

$$j = 1 + (1 + d) + (1 + 2d) + (1 + 3d) + (+ 4d) + (1 + 5d) \ldots + [1 + (n - 1)d] = n + d\ [1 + 2 + 3 + 4 + 5 \ldots (n - 1)]$$

from which it follows that $j = n + 2\ \frac{n(n - 1)}{2}$.

Now $\frac{n(n - 1)}{2}$,

the sum of the (n - 1) first numbers starting from unity, is the (n - 1)th triangular number; we have then these two theorems:

1. *The nth polygonal number equals n, plus d times the (n - 1)th triangular number, (d) being the rate of the progression of the gnomons;*

2. *The polygonal numbers from the same row n, form a progression by difference, of which the first term is n, and of which the rate is the (n - 1)th triangular number.*

The triangular numbers are so named because, if they are arranged one below the other with the addition of the gnomons decomposed into units, triangular figures result. (See I, XIX).

The square numbers are so named because a square form can be given to the groups of unit-marks of which these numbers are composed. (See I, XXV).

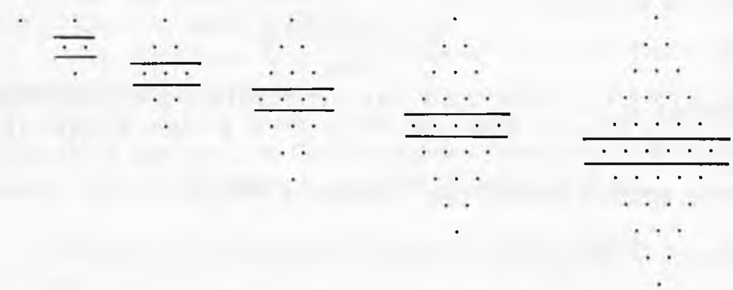
The figure of the square numbers can also be obtained by the formula

$$j = n + d\,\frac{n(n-1)}{2} \text{ which, for } d = 2, \text{ becomes}$$

$$j = n + 2\,\frac{n(n-1)}{2}.$$

We shall write on a line the n units of the number n, then place on either side of this line the unity-marks of which the triangular number (n - 1)st is composed. We will obtain the following quadrangular figures (replacing the unity-marks with points):

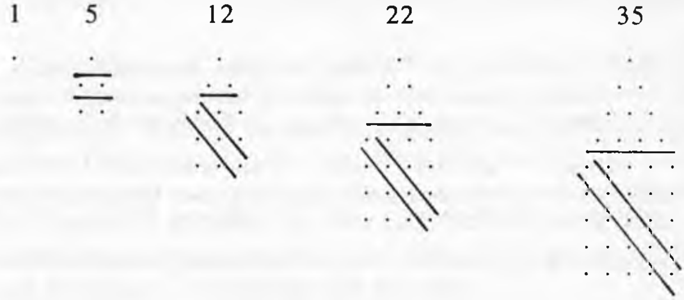
1 4 9 16 25 36



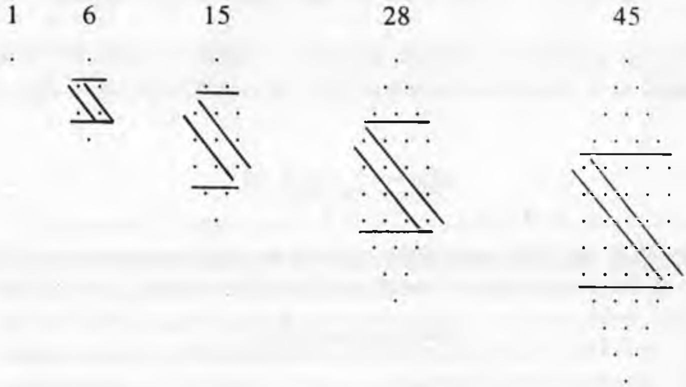
The pentagonal numbers are given by the formula

$$j = n + 3\,\frac{n(n-1)}{2}.$$

We will then obtain their representation by adding to n, three times the (n - 1)st triangular number; they can therefore be figured in the following manner:



We will have hexagonal numbers by adding to n four times the (n - 1)st triangular number; they can therefore be given in this form:



It can be noted that the natural series of hexagonal numbers is equal to the series of triangular numbers of every other row. It can, in fact, be demonstrated that the nth hexagonal number is equal to the (2n - 1)st triangular number, for according to the general formula (A) given above, each of them equals n (2n - 1).

Another thing to be noted is that the *perfect numbers*, that is to say those equal to the sum of their aliquot parts, are all *hexagonal and consequently triangular.*

Indeed the nth hexagonal number $j = n\ (2n - 1)$. Let us suppose that $n = 2k$. We will have $j = 2^k\ (2^{k+1} - 1)$. This is the formula which gives the perfect numbers when the factor $2^{k+1} - 1$ is first; therefore the perfect numbers are hexagonal and consequently triangular. Thus

6 =	2 x 3	is the	2nd hexag. and the 3rd triang.
28 =	4 x 7	is the	4th hexag. and the 7th triang.
496 =	16 x 31	is the	16th hexag. and the 31 st triang.
8,128 =	64 x 127	is the	64th hexag. and the 127th triang.
33,550,336 =	4,096 x 8,191	is the	4,096 hexag. and the 8,191st triang.
8,589,869,056 =	65,536 x 131,071	is the	65,536th hexag. and the 131,071st triang.
137,438,691,328 =	262,144 x 524,287	is the	262,144th hexag. and the 524,287th triang.

Note VI. — *On the Pyramidal numbers* (I, XXX).

The nth pyramidal number, having a triangular base, is the sum of the first n triangular numbers. It is demonstrated to be equal to

$$\frac{n(n+1)\ (n+2)}{1\ .\ 2\ .\ 3}$$

Likewise, the nth pyramidal number, having a square base, is the sum of the first n square numbers. It can be shown to be equal to

$$\frac{n(n+1)(2n+1)}{1\ .\ 2\ .\ 3}$$

The truncated pyramidal number is obtained by evaluating the total pyramid and that which has been taken off and taking the difference between the two values. If we have a triangular truncated pyramid whose lower base side has the value of n and the upper base p, the truncated pyramidal number will have the value of

$$\frac{n(n+1)(n+2) - (p-1)\,p(p+1)}{1\ .\ 2\ .\ 3}$$

Note VII. — *On Lateral and Diagonal Numbers.* (I, XXXI).

The lateral and diagonal numbers are defined by their generation. Theon explains it thus: he first takes the side 1 and the diagonal 1, then he successively determines the other sides by adding the diagonal to the preceding side, and he determines the other diagonal numbers by adding twice the corresponding side to the preceding diagonal. By this rule, the following table is obtained, completed by the addition of the double square of the sides and the squares of the diagonal numbers:

sides	diagonal numbers	double sqr. of the sides	square of the diagonal numbers
1	1	2	1 = 2 - 1
2 = 1 + 1	3 = 1 + 1 x 2	8	9 = 8 + 1
5 = 2 + 3	7 = 3 + 2 x 2	50	49 = 50 - 1
12 = 5 + 7	17 = 7 + 5 x 2	288	289 = 288 + 1
29 = 12 + 17	41 = 17 + 12 x 2	1682	1681 = 1682 - 1
70 = 29 + 41	99 = 41 + 29 x 2	9800	9801 = 9800 + 1
169 = 70 + 99	239 = 99 + 70 x 2	57122	57121 = 57122 - 1
			etc

This rule of Theon gives, in whole numbers, the resolution of the right isosceles triangle, with this condition, that the difference between the square of the hypotenuse and the double square of the side of the right angle is only one unit, that is to say, that it gives in whole numbers the solutions of the equation $y^2 - 2x^2 = \pm 1$. Let us suppose that $y = a$ and $x = b$ as a solution to the equation, that is to say, that we have $a^2 - 2b^2 = \pm 1$. I say that $x' = b + a$ and $y' = a + 2b$ are also a solution of it. It can indeed be deduced that these two last relations $y'^2 - 2x'^2 = 2b^2 - a^2$. Now $a^2 - 2b^2 = \pm 1$ by hypothesis, therefore $y'^2 - 2x'^2 = \pm 1$. But $y = x = 1$ is a first solution to the equation $y^2 - 2x^2 = -1$. Therefore, according to the rule given by Theon, an infinity of other solutions can be concluded.

Note VIII. — *On the Perfection of the Number Ten* (I, XXXXII).
(*The tetractys*)

The number 10 = 1 + 2 + 3 + 4. Now 1 was the principle of numbers; 2 represented the first line (the straight line which is defined by two of its points); 3 represented the first surface (the triangle defined by its three apexes); and 4 represented the first solid (the tetrahedron defined by its four apexes). Therefore, the decad 1 + 2 + 3 + 4 symbolized all that exists.

Note IX. — *On the Addition and Subtraction of the Consonances* (II, XIII,)

Let us call A, B, and C three sounds such that the interval from B to A is, for example, a fifth and the interval from C to B a fourth. We will call the numbers corresponding to these three sounds as A, B and C. C is $^4/_3$ rds of B, and B is $^3/_2$ nds of A, therefore C is $^4/_3$ rds of $^3/_2$ nds of A, in other words, C = 2A. Even though the interval of C to A is the product of the two intervals which it includes, it is said to be the sum of these two intervals: thus the octave is said to be the sum of a fifth and a fourth, but the number which measures the octave is the product of the two numbers which measure the fourth and the fifth.

Let us again call A, B, C, three sounds such that the interval from A to B is, for example, a fourth, and the interval of C to A is a fifth. We shall take a, b, c as the numbers corresponding to these sounds and x as the interval between c and b. According to the preceding remark, we have $^3/_2 = {}^4/_3\,x$, from which it follows that $x = {}^3/_2 : {}^4/_3 = {}^9/_8$ = one tone. Although the interval of C to B is the quotient of the interval of A to C by the interval of B to A, it is said that it is the difference between these two intervals. Thus it is said that the tone is the excess of the fifth over the fourth, but the number which measures the tone is the quotient of the two numbers which measure the fifth and the fourth.

Note X. — *The musical diagram of Plato contains four octaves, a fifth and a tone*. (II, XIII, 1)

Plato's musical diagram indeed contains the sounds corresponding to the terms of the two progressions 1, 2, 4, 8 and 1, 3, 9, 27 and stops at 27. The first sound of the first octave being represented by 1, the first sounds of the second, the third, the fourth and the fifth octaves are respectively represented by 2, 4, 8, 16. The fifth of this fifth octave is expressed by $16 \times {}^3/_2 = 24$. In order to add a tone to this fifth, he multiplies 24 by $^9/_8$. The result is 27, the last term of Plato's diagram, which contains consequently four octaves plus a fifth and a tone. [5]

Note XI. — *On the Value of the Half-Tone* (II, XIV).

Half of the tone $1 + {}^1/_8$ is not $1 + {}^1/_{16}$. This half x is given by the equation $x^2 = {}^9/_8$ from which it follows that $x = \sqrt{^9/_8}$. But it must be noted that the value of $1 + {}^1/_{16} = {}^{17}/_{16}$ is closely approximated, for in calculating the square one obtains $^{289}/_{256}$ which differs only by $^1/_{256}$ from the tone

$$^9/_8 = \frac{9 \times 32}{8 \times 32} = \frac{288}{256}$$

The leimma is less than a half-tone, because we have, as it is easy to verify:

$$\left(\frac{256}{243}\right)^2 < \frac{9}{8} \quad \text{therefore} \quad \frac{256}{243} < \sqrt{\frac{9}{8}}$$

Note XII. — *On the Perfect Musical System formed of Two Octaves.* (II, XXXV)

The musical scale of the ancient Greeks, described by Theon, encompassed an extension of the human voice. It was a descending series of two octaves. It was formed of four small systems each composed of four sounds, the extremes of which gave the fourth, the governing consonance from which the others were derived (II, XIII, 1.)

These small systems were called tetracords because the sounds were given by the four-stringed lyre. The strings of the instrument and the sounds that they made bore the same name. The two ex-

[5] Cf. *Timaeus, 34 d—35 d*

tremes of each tetracord were invariables or unmoving; the two intermediary ones were variable or mobile, they received different degrees of tension constituting three principal types of harmony: the diatonic, the chromatic and the enharmonic.

The first tetracord was called the tetracord of the upper tones or hyperboles.

The second was called the disjunct tetrachord or tetrachord of the *disjuncts,* because its last chord, that is to say, the lowest, was separated from the first or highest of the following tetrachord; it differed from it by one tone. The two first tetrachords had a common string; the lowest string of the tetrachord of the hyperboles was at the same time the highest string of the tetrachord of the disjuncts.

The third was the mean tetracord, or tetracord of the *meses.*

. The fourth was called the tetracord of the lows or the *hypates.* These two tetracords had a common string: the lowest string of the tetracord of the meses was at the same time the highest of the tetracord of the hypates.

The first and the second tetracord having a common string, as well as the third and the fourth, the whole of four tetracords made only fourteen sounds. In order to complete the two octaves, a fifteenth sound was added below the lowest sound of the tetracord of the hypates, which was lower by one tone and which was called the proslambanomenos, which is implying an addition, that is to say, "it is an added sound", or "added string."

In the same way that the tetracords were designed by names relating to their position in the musical scale, the strings were designated by names relating to their position in each tetracord.

The highest was the *nete* of the hyperboles or the extreme. [6]

The second was the *paranete,* that is to say, the neighbor of the nete.

The third was called the *trite* of the hyperboles, or third.

The fourth and the fifth were the *nete* and the *paranete* of the disjuncts, and

The sixth, called the *trite* of the disjuncts which, was the third string of the disjunct tetrachord.

The seventh and eighth were the *paramese* παράμεσος or the neighbor of the mese, and the *mese.*

The ninth was the *lichanos* of the mese [7]

[6] Νηζη *for* Νεαζη, *from most extreme, which is at the extremity.*

[7] Λιχανος *indicator, from index finger: the lichanos indicated the type which was diatonic, chromatic or enharmonic, according to whether the interval from the sound of this string to the sound of the preceding string had the value of a tone, a tone and a half, or two tones.*

The tenth and eleventh were the *parhypate* and the *hypate* of the meses Παρυπάτη and ὑπάτη.

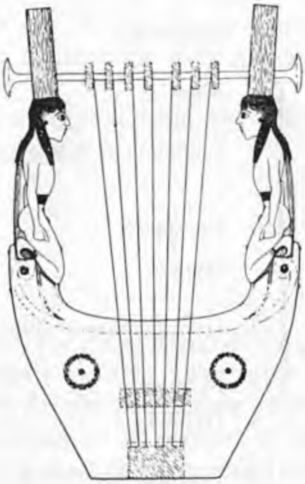
The twelfth was the hyperhypate, ὑπερυπάτη or lichanos of the hypates. λίχανος ὑπάτων.

The thirteenth and the fourteenth were the *parhypate* of the hypate and the *hypate* of the hypates,

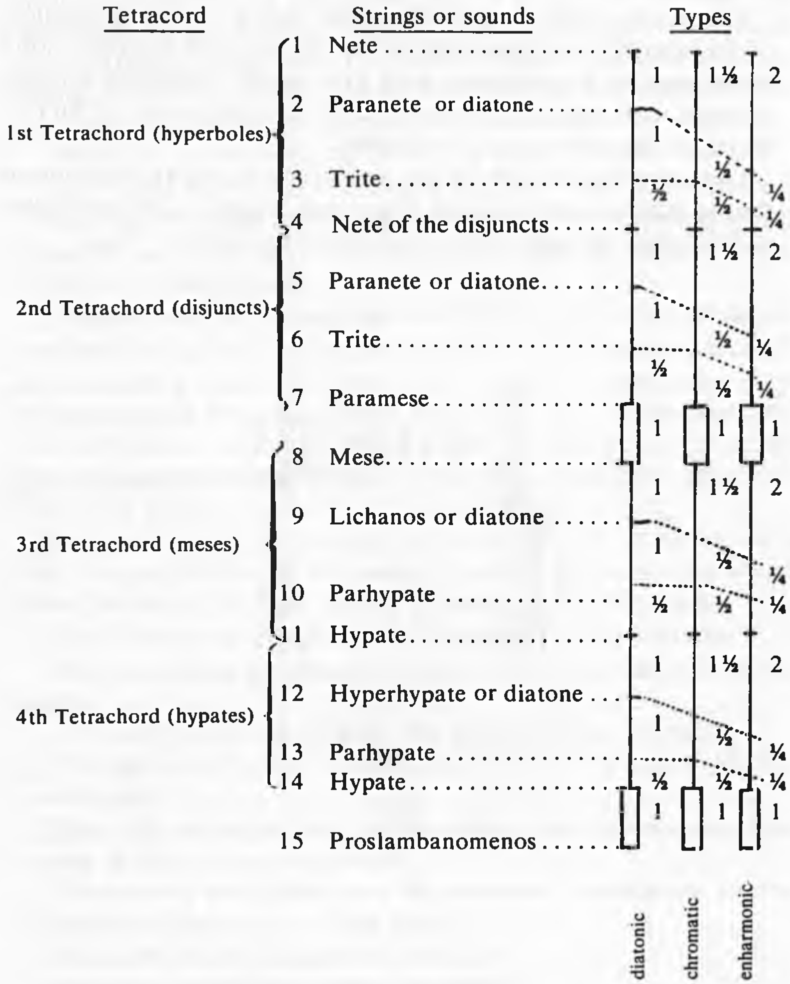
Finally the fifteenth was the proslambanomenos.

The second string of each tetrachord, that is to say the paranete of the hyperboles, the paranete of the disjuncts, the lichanos of the meses and the hyperhypate, were also called, according to type; diatone, chromatic, or enharmonic, of the hyperboles, of the disjuncts, of the meses or of the hypates.

The following is a table of this perfect system, with indications of the successive intervals in the three types, diatonic, chromatic and enharmonic, the half-tone or leimma being equal to $^{256}/_{243}$.



The Perfect System, formed of two octaves, containing of three types: diatonic, chromatic, enharmonic.



(The conjunct tetrachord which Theon discusses on pp. 60-61 is bounded by strings 5 and 8.)

Note XIII. — *The Musical Diagram of Plato* (II, XXXVI)

The Probably Intentional Error of Timaeus of Locris.

Plato, in the *Timaeus,* in order to explain the formation of the soul of the world, admits that God first divided the essence into *seven* parts which were related to one another as the terms of the two progressions 1, 2, 4, 8 and 1, 3, 9, 27, the first of which having the ratio of 2 and the other the ratio of 3.

He then says that God inserted two means between the successive terms of these two progressions, of which one, which we call the arithmetic mean, equals their half-sum, and the other is such that it is greater than one extreme and less than the other by the same fraction of the extremes, that is to say, if x is the mean inserted between a and b, then $x - a : b - x = a : b$, therefore

$$x = \frac{2ab}{a + b} = \frac{ab}{\frac{1}{2}(a+b)}$$

so that this mean between the two numbers is obtained by dividing the double product of these two numbers by their sum, or the product of the two numbers by their half-sum. This is called the *harmonic mean.*

By means of this double insertion, the following numbers are obtained (read horizontally):

1	$\frac{4}{3}$	$\frac{3}{2}$	2
2	$\frac{8}{3}$	3	4
4	$\frac{16}{3}$	6	8

In this progression, the relationship of the arithmetic mean to the harmonic mean equals $^{9}/_{8}$: this is the value of the tone.

Plato next inserts between each term of the double progression and the harmonic mean which follows it, as well as between the arithmetic mean and the following term, two terms such that the relationship of each of them to the preceding is also $^{9}/_{8}$ ths.

This operation is made on the progression 1, 2, 4, 8 and continued until arriving at the term 27, gives the results contained in the following table:

TABLE I
(to be read vertically)

	1	2	4	8	16
	$9/8$	$9/4$	$9/2$	9	18
	$81/64$	$81/32$	$81/16$	$81/8$	$81/4$
Harmonic means	$4/3$	$8/3$	* $16/3$	* $32/3$	* $64/3$
Arithmetic means	$3/2$	3	6	12	24
	$27/16$	$27/8$	$27/4$	$27/2$	27
	$243/128$	$243/64$	$243/32$	$243/16$	
	2	4	8	* 16	

In order to substitute whole numbers for these which are mostly fractions, they can be reduced to the smallest common denominator, 128 x 3 = 384, and all multiplied by this denominator, thus obtaining the following table:

TABLE II

	384	768	1536	3072	6144
	432	864	1728	3456	6912
	486	972	1944	3888	7776
	512	1024	*2048	*4096	*8192
	576	1152	2304	4608	9216
	648	1296	2592	5184	10368
	729	1458	2916	5832	
	768	1536	3072	*6144	
Sums	4535	8302	16604		
Total		29441			

If one likewise inserts a harmonic mean and an arithmetic mean between the successive terms of the triple progression, these numbers will be obtained (to be read horizontally):

1	$3/2$	2	3
3	$9/2$	6	9
9	$27/2$	18	27

The intervals of 1 to 3, 3 to 9 and of 9 to 27 being those of the octave and the fifth, Proclus [8] admits that Plato first filled the interval of 1 to 3, like those of the double progression, and that he then tripled the terms obtained from 1 to 3, in order to have those of 3 to 9, and tripled the terms of 3 to 9 in order to have those of 9 to 27.

The operation thus effectuated gives the results which can be multiplied by 128 x 3 = 384, the smallest common multiple of the denominators, in order to substitute whole number proportions for them. Thus the two following tables are obtained:

Table III *Table IV*

(to be read vertically)

	1	3	9	384	1152	3456
		—	—		—	—
	9/8	27/8	81/8	432	1296	3888
	81/64	243/64	* 729/64	486	1458	*4374
Harmonic	4/3	4	12	512	1536	4608
means	[3/2	9/2	27/2	576	1728	5184
	27/16	81/16	243/16	648	1944	5832
	243/128	* 729/128	* 2187/128	729	*2187	*6561
Arithmetic	[2	6	18	768	2304	6912
means	9/4	27/4	81/4	864	2592	7776
	81/32	243/32	* 729/32	972	2916	8748
	8/3	8	24	1024	3072	9216
	3	9	27	1152	3456	10368
					Sum	76923

We've placed an asterisk beside the terms of the triple progression (Tables III and IV) which are not part of the progression of doubles, and the terms of the double progression (Tables I and II) which are not part of the progression of triples.

In the treatise *On the Soul of the World and on Nature* which bears the name Timaeus of Locris, we find the following: (end of Chapter I).

[8] *Proclus in Timaeum*, p. 193 and the following from the edition de Bale, 1534.

> "God made the soul first, by taking in the mixture which he had formed, a part equal to 384 units. This first number found, it is easy to calculate the terms of the double progression and of the triple progression. All these terms arranged according to the intervals of tones and half-tones, are 36 in number and give a total sum equal to 114,695; and the divisions of the soul are themselves 114,695 in number."

Now the evident intention of Plato was to not go beyond 8 in the progression of doubles and 27 in the progression of triples. Therefore his diagram contains:

1. 22 terms of the double progression, comprised of 1 x 384 to 8 x 384 that is to say, from 384 to 3072 (Table II), which have the value of 29,441

2. 1 term of the triple progression including, from 384 to 3072, which is not part of the double progression (see Tables IV and II), which is 2,187

and finally,

3. 12 terms of the triple progression, comprised of 9 x 384 to 27 x 384, that is to say, from 3456 to 10368 (Table IV) which have the value of 76,923

Sum. 108,551

Thus, the diagram of Plato contains (22+1+12) or 35 different terms (and not 36), and the sum of these 35 terms is 108,551 and not 114,695. The difference (6,144) of the two results is the term 16 x 384 of the progression of doubles (Table II), a term which he does not take into account, since it is greater than 8 x 384 in the progression of doubles and does not take part in the triple progression. If it is counted it is necessary to count also the two terms 4096 and 8192 of the double progression, which do not take part in the triple progression.

There is then an error in the treatise bearing the name of Timaeus of Locris. If, following the intentions of Plato, one does not go further in making the insertions than the cubes of 8 and 27 in the respective progressions 1, 2, 4, 8 and 1, 3, 9, 27, the musical diagram of Plato contains 35 terms, the sum of which is 108, 551, less by 6144 than the sum of 114,695 given by Timaeus of Locris.[9]

[9] The error of Timaeus of Locris is reproduced by all the commentators. See Abbe Roussier, *Memoir sur la Musique des anciens*, Paris — 1770, in 4^{e}, p. 248 onwards; V. Cousin, Translation of the *Oeuvres de Platon*, Paris 1839, in 8^{e}, t. XII, p. 335; J. Simon, *Du commentaire du Timée de Platon* par Proclus, Paris 1839, in -8^{e}, p. 163; A.-J.-H. Vincent, *Notices et extraits des manuscrits de la Bibliothèque Royale*, 1847, t. XVI, 2^{e} partie, p. 176 onwards etc.

Knowing in what veneration the Pythagoreans held the tetraktys, we firmly believe that the error of the Pseudo-Timaeus was not intentional. The number 35 was certainly endowed with perfection; it was the product of the septenary number by the half-decad, but the number 36 was still more perfect, being the product of the first even square by the first odd square; and consequently, it was itself a square, that is to say, a harmony, and furthermore its side 6 was a truly perfect number, in other words equal to the sum of its aliquot parts, for $6 = 1 + 2 + 3$. The number 36 had another virtue if we listen to Plutarch: "...This tetraktys, that is 36, celebrated by the Pythagoreans, seems to have the admirable quality that it is the sum of the first four even numbers and the first four odd numbers. . . ."

$$(1 + 3 + 5 + 7) + (2 + 4 + 6 + 8) = 16 + 20 = 36$$

(from; *The Creation of the Soul*, in the *Timaeus*, xxx.)

While the Pythagorean philosophers wanted to find quaternaries everywhere, Timaeus, in order to complete the great quaternary 36, would have added the term 6144 corresponding to the sound 16, the octave of the sound 8 which is the last term of the progression 1, 2, 3, 4, to the 35 terms of Plato's musical diagram.

If the Pseudo-Timaeus did not also add to Plato's diagram the two terms 4096 and 8192, which are respectively the highest fifth and the lowest fourth of 6144 and which, like 6144, do not constitute part of the terms inserted in the progression of triples, it is because then the total number of terms would have been 38 instead of 36.

Note XIV. — *Why the Number Six was called that of Marriage* (II, XLV)

It was also called marriage because it is the product of the first even number 2 by the first odd number 3. The odd numbers were considered as male and the even numbers as female. "If each of them are divided into units," says Plutarch (translation of Amyot), "the even will show an empty place at the middle, there where the odd always has the middle filled with one of its parts, and for this reason, they (the Pythagoreans) have the opinion that the even has more resemblance to the female and the odd to the male:" (Questions romaines, CII, p. 288)

Note XV. — *On the Euripes* (II, XLVI).

The name euripes is given to the currents which are produced in channels or straits (narrows).

The most famous was that of Chalcis, between the island of Evea and Boeotia, whose direction changed seven times per day, according to most ancient authors: "There are particular tides in certain places," says Pliny, "thus the flux occurs several times in the strait of Messina, at Tauromenius, and seven times per day in the Euripe near Evea." (*Natural History* II, c) [10]

The commentator of Stobea rightly attributed the alternative movements of the euripe of Chalcis to the effort of the waves to surmount the channel. (See Eclogae physicae, t. II, p 447, ed. Heeren, article entitled: "On the Ebb-Tide of Evea". The effect of this phenomenon can be best observed from the waterfront of the town of Chalkis, where a draw-bridge connects the island of Evea to the mainland.)

The variations of the flux of the euripes were very irregular, and this inconsistency was very well known.

Plato says in the *Phaedo,* "Neither in things nor in argument is there anything true and stable; but all is in continual ebb and flow, like the euripe, and nothing remains for a moment in the same state..." (*Phaedo*, XXXIX, 90 c)

Lucian also says in the *Pharsalia*: "the inconstant tides of the euripe drag the vessels of Chalcis towards Aulis, so deadly to boatmen."[11] — (*La Pharsalia,* song V, vs. 235-236)

The superstitious idea attached to the number *seven* appears to explain Theon's hypothesis, that the euripes vary seven times per day.[12]

[10] This island can be reached by car from Athens after a two-hour drive.

[11] It is indeed impossible to control a row-boat at the moment the ebb-tide changes direction while in the straits of Evea.

[12] A fact verified now as it happens this present day and anyone can observe. (Toulis)

Note XVI. — *The determination of the harmonic mean between two numbers.* (II, LXI).

a, b, c, being the three numbers which give the harmonic proportion a - b : b - c = a : c, the first rule of Theon, translated by the formula

$$b = \frac{(a - c)c}{a + c} + c, \text{ a value equal to } \frac{2ac}{a+c},$$

It is therefore general, whatever be the relationship of a to c.

The second rule is translated by the formula

$$b = \frac{(a - c)^2}{2(a + c)} + c; \text{ this value is equal to } \frac{2ac}{a+c}$$

only when a = c, a solution to be rejected, and when a = 3c. Theon actually gives the second rule for the numbers in triple relationship, 18 and 6.

The author having made the remark (II, LVII) that in the harmonic proportion, the product of the sum of the extreme numbers by the harmonic mean is equal to the double product of the extreme numbers, we are surprised that he did not conclude from this equality the value of the harmonic mean.

Note XVII. — *On the Measure of the Earth's Volume* (III, III)

The passage is altered and the manuscripts have a blank gap at the end. Henri Martin, in trying to reconstitute it, made an error of calculation. The diameter d of the earth being equal to 80,182 stades, we have

$$d^2 = 6{,}429{,}153{,}124 \text{ instead of } 6{,}427{,}153{,}124.$$

The incorrect number 7, of the hundreds of thousands, substituted for the correct figure 9, gave H. Martin inexact values for d^3, for $^1/_{14}\,d^3$ and for $^{22}/_3$ of $^1/_{14}\,d^3$, which expresses the volume of the sphere from the diameter d. It is necessary that

$$d^3 = 515{,}502{,}355{,}788{,}568 \text{ instead of } 515{,}341{,}991{,}788{,}568$$

$$^1/_{14}\,d^3 = 36{,}821{,}596{,}842{,}040 + {^4/_7} \text{ instead of } 36{,}810{,}142{,}270{,}612$$

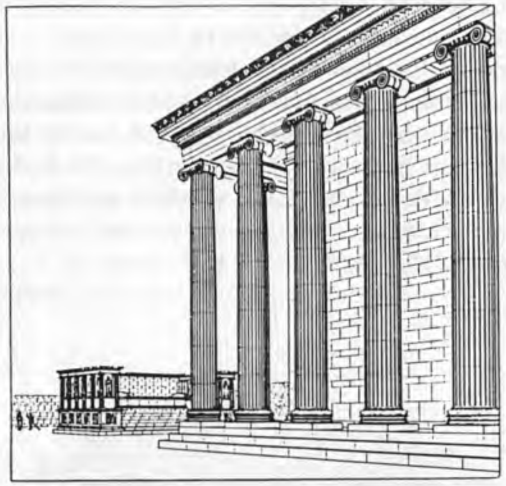
$$^{11}/_{21}\,d^3 = 270{,}025{,}043{,}508{,}297 + {^{11}/_{21}} \text{ instead of } 269{,}941{,}043{,}317{,}821 + {^1/_3}$$

Thus the volume of the earth, evaluated in cubic stades, supposing that the relationship of the circumference to the diameter is equal to $^{22}/_7$, has the value of 270 third myriads, 250 second myriads, 4,350 myriads, 8,297 monads and $^{11}/_{21}$, according to the measure of the arc of meridian made by Erathosthenes. Not only this fraction is illusory, but one can count, at the most, on the two or three first numbers of the result. It is the expression of this volume that we have restated, pp85 and 86 (of text).

Note XVIII. — *On the Myth of Pamphylian in Plato's Republic,* 616 B — 617 B. (III, XVI).

It results from Plato's tale that of the eight concentric globes, the first exterior one is that of the fixed stars, the second is that which bears Saturn, the third carries Jupiter, the fourth Mars, the fifth Hermes, the sixth Venus, the seventh the Sun, and the eighth the moon. The earth is at the center of the system.

The colors and speeds of the spindle-wheels correspond to those of the stars which they carry, and the unequal widths of the colored edges correspond to the unequal separation of the planets in their course through the zodiac and sometimes beyond. The sphere of the fixed stars is indeed of varied color, since the stars have diverse nuances. The seventh circle, that of the sun, is very brilliant; the eighth, that of the moon borrows its light from it. The slightly yellow shade of the second and fifth is also that of Saturn and Hermes. The whiteness of the third and the redness of the fourth perfectly characterize the aspects of Jupiter and of Mars. Finally, the sixth circle is given as the most brilliant following the sun, which is true of Venus. An equal speed is attributed to Hermes, Venus and the Sun: the Sun, in its apparent course around the earth, indeed pulls along Mercury and Venus.



INDEX

SUPPLEMENTAL READING

(Anonymous). L'Architecture Naturelle. Paris, 1946. (50 copies in folio) Translated from Latin mss, copy in Bibliotheque Nationale, Paris. 415 pages. *extraordinary*.

Brunés, Tons. Secrets of Ancient Geometry. 2 vols. International Science Pubs, Copenhagen, 1967.

Burgess, E. & Whitney, W. D. (translators) Surya Siddhanta, a Text Book of Hindu Astronomy. New Haven, 1860; rpr Wizards Bookshelf, 1978.

Clark, Walter E. The Aryabhatiya of Aryabhata. U. of Chicago Press, Chicago, 1930, (translation of technical astronomical work of 499 A.D.)

Dee, John. Monas Hieroglyphica. Gulielmus Silvius, Antwerp, 1564; Weiser, NYC, 1975.

______. The Mathematical Preaface. 1570. rpr Watson Neal Academic, NYC, 1975.

Guthrie, Kenneth Sylvan. Pythagoras Source Book and Library. Platonist Press, North Yonkers, N.Y. 1919.

Heath, Thomas L. Greek Astronomy. London, 1932; rpr AMS Press, NYC, 1969.

Kircher, Athanasius. Musurgia universalis, sive ars magna consoni et dissoni, Rome, 1650.

______. Liber diacriticus musurgia antiquo-moderna in qua tam de varia utriusque musicae ratione disputatur; — Liber philologicus de sono artificioso sive musica, ejusque prima institutione, aetate, vicissitudine, propagatione. (in, Thes. antiq. sacr., v.32 1767) (Kircher wrote 23 incredible works . . . the above but a sample.)

McClain, Ernest G. The Pythagorean Plato. Nicholas Hays Ltd, NYC, 1978.

______. The Myth of Invariance. Shambhala, Boulder, 1978.

Oliver, George. Pythagorean triangle. London, 1875; rpr Wizards Bookshelf, 1975.

Scott, Cyril. Music: Its Secret Influence Throughout the Ages. 1933; rpr Weiser, NYC, 1973.

Schwaller de Lubicz, R. A. The Temple in Man. Tr by Robert Lawlor from French. Autumn Press, Brookline, 1977.

______. The Temple Of Man, 3 vols, tr by Robert Lawlor, Autumn Press, Brookline, 1979.

Skinner, J. Ralston. Key to the Hebrew Egyptian Mystery in the Source of Measures. NYC, 1875; Supplement, 1876; Cincinnati, w/suppl., 1894 (all distroyed by fire); Philadelphia, 500 copies, 1937; Minneapolis, 535 copies, 1972; & 1975.

Taylor, C. A. Physics of Musical Sounds. Crane-Russak Co., NYC, 1965.

Taylor, Thomas. (1758-1835) Theoretic Arithmetic, etc, London, pvt printing, 1816; rpr Phoenix Press (Manly Hall) Los Angeles, 1934; rpr Weiser, NYC, 1976.

______. Medicina Mentis: Or, a Specimen of Theological Arithmetic. Manuscript of 21 pages, copy in Houghton Library, Harvard U. Published in quarterly periodical *Shrine of Wisdom*, London, 194?

Wake, C. Staniland. Musical Tone and Color. in, *Music*, (monthly periodical) Chicago, 1896. (see also: National Union Catalog, for over 40 more such listings)

Index

Authors and principal material cited in Theon

The first number, in Roman numerals, indicates the book of Theon, the second in ordinary numerals indicates the section.